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Efficient Quadrature Rules for Numerical Integration Based on Linear Legendre Multi-Wavelets

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Abstract. In this work, generalize solution based on linear Legendre multi-wavelets are proposed for single, double and triple integrals with variable limits. To obtain the numerical approximations for integrals, an algorithm with the properties of linear Legendre multi-wavelets are applied. The main benefits of this method are its simple applicable and efficient. Furthermore, the error analysis for single, double and triple integrals is worked out to show the efficiency of the method. Numerical examples for the integrals are conducted by using linear Legendre multi-wavelets in order to validate the error estimation.

1. Introduction

Numerical integration has several applications in science and engineering. There are many engineering problems require the evaluation of the integral such as skin friction coefficient for the governing boundary layer equations in fluid dynamic. To determine the probability of electron in a region of space by solving the Schrodinger equation in quantum mechanics and several other problems. A lot of research have been done to solve numerical integration problems in terms of quadrature rule such as Newton-Cotes formulas and Gauss quadrature [1, 2, 3, 4, 5, 6, 7]. Regardless of the simplicity of quadrature rule, there exists a few disadvantages. For example, the Newton-Cotes formulas cause erratic behavior with high degree polynomial interpolation when the equally spaced nodes are large. The Gaussian quadrature rule is derived by method of undetermined coefficients but the resulting equations for the nodes and weights are nonlinear. This procedure is complicated to find the nodes and weights. Other than that, there is limitation with Gaussian quadrature rule method where the limits of integration need to convert into -1 to 1 [7]. To overcome the disadvantages, a new method based on wavelets approximation is propose to find the numerical solutions of integrals. There are several types of wavelets have been used in numerical approximations, for examples, Daubechies' [8], Chebyshev wavelets [9] and Haar wavelets [10, 11, 12, 13].

According to [11], Haar wavelets are used to solve for single, double and triple integrals with variable limits. Algorithm based on Haar wavelets have been obtained to approximate the integrals for single,



double and triple integrals. Linear Legendre multi wavelets display similar properties to the Haar wavelets. Due to this similarities, the linear Legendre multi-wavelets should also be able to solve the multi-dimensional integral easily [14]. There are numerous research have been done to solve problems such as integral equations by using linear Legendre multi-wavelets [15, 16].

In this work, we deduce generalized solution based on linear Legendre multi-wavelets. The organization of this paper is as follows. We introduce about linear Legendre multi-wavelets in section 2, and in section 3 numerical system by linear Legendre multi-wavelets are shown for single, double and triple integrals with variable limits. Error analysis are describe in section 4 to show the convergence of the method and numerical results are reported in section 5. Finally further discussion and conclusion are drawn in section 6 and 7.

2. Linear Legendre multi-wavelets (LLMW)

In this paper, we used LLMW to handle with the problems of single, double and triple integrals with variable limits. Generally, wavelets have been used in various fields of engineering and science. It consists of two functions which are scaling function and mother wavelet. To construct the linear Legendre mother wavelets, we first introduce the scaling functions. There are two scaling functions for linear Legendre multi-wavelets which are as follows:

$$\phi_0(x) = 1, \quad \phi_1(x) = \sqrt{3}(2x - 1), \quad a \leq x < b.$$

We let $\psi^0(x)$ and $\psi^1(x)$ as the corresponding mother wavelets, then by multiresolution of analysis (MRA) we have

$$\begin{aligned} \psi^0(x) &= a_0\phi_0(2x) + a_1\phi_1(2x) + a_2\phi_0(2x - 1) + a_3\phi_1(2x - 1), \\ \psi^1(x) &= b_0\phi_0(2x) + b_1\phi_1(2x) + b_2\phi_0(2x - 1) + b_3\phi_1(2x - 1) \end{aligned}$$

and there are suitable conditions that applied on $\psi^0(x)$ and $\psi^1(x)$. As a result, the formulas for linear Legendre mother wavelets are obtained as

$$\psi^0(x) = \begin{cases} -\sqrt{3}(4x - 1), & \left[a, \frac{a+b}{2} \right) \\ \sqrt{3}(4x - 3), & \left[\frac{a+b}{2}, b \right) \end{cases}, \quad \psi^1(x) = \begin{cases} 6x - 1, & \left[a, \frac{a+b}{2} \right) \\ 6x - 5, & \left[\frac{a+b}{2}, b \right) \end{cases},$$

By translating and dilating the mother wavelets, the linear Legendre multi-wavelets is constructed as

$$\psi_{kn}^j(x) = \begin{cases} 2^{\frac{k}{2}} \psi^j \left(2^k \frac{x-a}{b-a} - n \right), & a + n \frac{(b-a)}{2^k} \leq x < a + (n+1) \frac{(b-a)}{2^k}, \\ 0, & \text{otherwise,} \end{cases}$$

here $k, \in \mathbb{Z}$, $n = 0, 1, \dots, 2^k - 1$, and $j = 0, 1$ are defined on the interval $[a, b)$.

Any function $f(x)$ in the interval $[a, b)$ can be written as

$$f(x) \approx c_0\phi_0(x) + c_1\phi_1(x) + \sum_{k=0}^M \sum_{j=0}^1 \sum_{n=0}^{2^k-1} c_{kn}^j \psi_{kn}^j(x) = C^T \Psi(x), \quad (1)$$

where C and $\Psi(x)$ are

$$C = [c_0, c_1, c_{00}^0, c_{00}^1, \dots, c_{M0}^0, c_{M1}^0, \dots, c_{M(2^M-1)}^0, c_{M0}^1, c_{M1}^1, \dots, c_{M(2^M-1)}^1]^T,$$

and

$$\Psi = [\phi_0, \phi_1, \psi_{00}^0, \psi_{00}^1, \dots, \psi_{M0}^0, \psi_{M1}^0, \dots, \psi_{M(2^M-1)}^0, \psi_{M0}^1, \psi_{M1}^1, \dots, \psi_{M(2^M-1)}^1]^T.$$

3. Numerical integration based on LLMW

In this section, we apply LLMW for integration of double and triple integrals with variable limits by according to [14].

3.1. Numerical system for single integral by LLMW

In [14], numerical integration formula by using LLMW were derived for single, double and triple integrals with definite limits. In this work, we apply LLMW for double and triple integrals with variable limits using the similar system derived for single integral with definite limits. The following equation is the numerical system derived in [14] for single integral with definite limits

$$\int_a^b f(x) dx \approx \frac{(b-a)}{K} \sum_{i=0}^{K-1} f\left(a + \frac{(b-a)(i+0.5)}{K}\right). \quad (2)$$

where $K = 2^{k_1+2}$ and k_1 is the level of resolution of the LLMW.

3.2. Numerical system for double integral with variable limits

Consider the double integral with variable limits as follows:

$$\int_c^d \int_{a(y)}^{b(y)} F(x, y) dx dy.$$

We apply equation (2) to the integral

$$\int_{a(y)}^{b(y)} F(x, y) dx.$$

while variable y is constant. Then, we obtain the following approximations.

$$\int_{a(y)}^{b(y)} F(x, y) dx \approx \frac{(b(y) - a(y))}{K} \sum_{i=0}^{K-1} F\left(a(y) + \frac{(b(y) - a(y))(i+0.5)}{K}, y\right) = G(y). \quad (3)$$

In the following, we get the numerical system for double integral with variable limits by using the same steps as in equation (2)

$$\int_c^d \int_{a(y)}^{b(y)} F(x, y) dx \approx \int_c^d G(y) dy \quad (4)$$

$$\approx \frac{(d-c)}{L} \sum_{j=0}^{L-1} F\left(c + \frac{(d-c)(j+0.5)}{L}\right). \quad (5)$$

where $L = 2^{k_2+2}$ and k_2 is the level of resolution of the LLMW.

3.3. Numerical system for triple integral with variable limits

By using the same way as previous numerical systems, we obtained the numerical integration formula for triple integrals with variable limits as below

$$\int_e^f \int_{c(z)}^{d(z)} \int_{a(y,z)}^{b(y,z)} F(x, y, z) dx dy dz \approx \frac{(f-e)}{P} \sum_{l=0}^{P-1} H\left(f + \frac{(f-e)(l+0.5)}{P}\right). \quad (6)$$

where $H(z)$ and $G(y, z)$ are

$$H(z) = \frac{(d(z) - c(z))}{L} \sum_{j=0}^{L-1} G \left(c(z) + \frac{(d(z) - c(z))(j + 0.5)}{L}, z \right).$$

$$G(y, z) = \frac{(b(y, z) - a(y, z))}{K} \sum_{i=0}^{K-1} F \left(a(y, z) + \frac{(b(y, z) - a(y, z))(i + 0.5)}{K}, y, z \right).$$

In the equation (6), $P = 2^{k_3+2}$ where k_3 is the level of resolution of the LLMW.

4. Error Analysis

We continued with the definition of the Holder classes of order $H^s[0, 1]$, $0 < s < 1$. The set of all continuous functions on $[0, 1]$, which satisfies the inequality :

$$|f(x) - f(y)| \leq L |x - y|^s, \quad L > 0, \forall x, y \in [0, 1].$$

Holder classes are medium spaces between $C[0, 1]$ and $C^1[0, 1]$ such that:

$$C^1[0, 1] \subset H^s[0, 1] \subset C[0, 1].$$

Consider $f(x) \in H^s[0, 1]$, $0 < s < 1$, then

$$\|f - f_M\|_{L_2[0,1]} \leq \frac{L^2}{4^{Ms}(4^s - 1)} \left(\frac{3}{16} + \frac{4^s}{9} \left[\frac{8 \left(\frac{2}{3}\right)^s - 3 + 3s}{(s+2)(s+1)} \right]^2 \right).$$

where

$$f_M(x) = c_0 \phi_0(x) + c_1 \phi_1(x) + \sum_{k=0}^{M-1} \sum_{j=0}^1 \sum_{n=0}^{2^k-1} c_{kn}^j \psi_{kn}^j(x), \quad M \in \mathbb{Z}^+,$$

refer to [17]. The error bound are inversely proportion to the level of resolution M of the LLMW and will establish better approximation if increase the value of M .

5. Numerical Examples

In this section, we applied LLMW for solving numerical integration problems for double and triple integrals. The results will show the efficiency of LLMW by comparing it with reported in [11]. In [11] Haar wavelets have been applied to approximate double and triple integrals with variable limits of integration. We solved all of the examples in [11] using LLMW numerically and compare the numerical results with Haar wavelets at the same level dilation (J_i is the dilation for Haar and value of k_i is the dilation for LLMW, $i = 1, 2, 3$) to validate the error estimation.

5.1. Test Problems

Example	Problem	Exact Solution
1.	$\int_0^1 \int_0^y e^{ x+y-1 } dx dy$	$-2 + e$
2.	$\int_0^{\frac{\pi}{4}} \int_0^{\sin y} \frac{1}{\sqrt{1-x^2}} dx dy$	$\frac{\pi^2}{32}$
3.	$\int \int_R (x+y)^{-\frac{1}{2}} dx dy$ $= \int_1^2 \int_{\frac{1}{5-3y}}^{\frac{1}{2}(y+3)} (x+y)^{-\frac{1}{2}} dx dy$ $+ \int_2^3 \int_{\frac{1}{y-3}}^{\frac{1}{2}(y+3)} (x+y)^{-\frac{1}{2}} dx dy$ $+ \int_3^4 \int_{\frac{1}{y-3}}^{\frac{1}{2}(y+3)} (x+y)^{-\frac{1}{2}} dx dy$	$\frac{2}{3}(2 - 7\sqrt{3} - 15\sqrt{5} + 20\sqrt{6})$
4.	$\int_0^\pi \int_0^z \int_0^{zy} \frac{1}{y} \sin(\frac{x}{y}) dx dy dz$	$\frac{1}{2}(4 + \pi^2)$

5.2. Numerical Results

LLMW	Absolute Errors	Haar	Absolute Errors
$k_1, k_2 = 3$	1.1857E-04	$J_1, J_2 = 3$	4.7400E-04
$k_1, k_2 = 4$	2.7875E-05	$J_1, J_2 = 4$	9.7811E-05
$k_1, k_2 = 5$	6.7701E-06	$J_1, J_2 = 5$	2.7875E-05
$k_1, k_2 = 6$	1.7065E-06	$J_1, J_2 = 6$	6.7700E-06

Table 1. Absolute errors of Example 1

LLMW	Absolute Errors	Haar	Absolute Errors
$k_1, k_2 = 3$	6.2347E-06	$J_1, J_2 = 3$	2.4839E-05
$k_1, k_2 = 4$	1.5604E-06	$J_1, J_2 = 4$	6.2348E-06
$k_1, k_2 = 5$	3.9030E-07	$J_1, J_2 = 5$	1.5601E-06
$k_1, k_2 = 6$	9.7300E-08	$J_1, J_2 = 6$	3.9030E-07

Table 2. Absolute errors of Example 2

LLMW	Absolute Errors	Haar	Absolute Errors
$k_1, k_2 = 3$	1.8930E-04	$J_1, J_2 = 3$	7.5503E-04
$k_1, k_2 = 4$	4.7363E-05	$J_1, J_2 = 4$	1.8930E-04
$k_1, k_2 = 5$	1.1852E-05	$J_1, J_2 = 5$	4.7365E-05
$k_1, k_2 = 6$	3.0528E-06	$J_1, J_2 = 6$	1.1855E-05

Table 3. Absolute errors of Example 3

LLMW	Absolute Errors	Haar	Absolute Errors
$k_1, k_2, k_3 = 3$	9.0291E-04	$J_1, J_2, J_3 = 3$	3.5959E-03
$k_1, k_2, k_3 = 4$	2.2597E-04	$J_1, J_2, J_3 = 4$	9.0291E-04
$k_1, k_2, k_3 = 5$	5.6516E-05	$J_1, J_2, J_3 = 5$	2.2598E-04
$k_1, k_2, k_3 = 6$	1.4127E-05	$J_1, J_2, J_3 = 6$	5.6507E-05

Table 4. Absolute errors of Example 4

6. Discussion

All the test problems in this work are from [11]. [11] approximate the double and triple integrals by using Haar wavelets $h_i(x)$ where $i = 1, 2, \dots, 2M$. They denote $M = 2^J$ and $J = 0, 1, 2, \dots$ is the maximum level of resolution of Haar wavelets see [11] equation (1). Regarding table (1) and (3) for absolute errors in [11] suggest letting $M = 3, 5$ or 6 , where else this research favours taking the value of M as an order of 2. Therefore Haar wavelets functions of order 2 ($M = 8, 16, \dots$) are consider to approximate the double and triple integrals. Concerning table (4), the problem are related to three dimensional case and the bsolute errors are equivalent to the previous work. Moreover the absolute

error between this two methods for Haar wavelets and LLMW functions are compared using the same order of dilation in all the test problems.

7. Conclusion

In this work, numerical integration based on LLMW has been applied to approximate the numerical examples for double and triple integrals with variable limits of integration. Generalized solution for LLMW are obtained to approximate the integrals. By analyzing the numerical results in terms of absolute errors between LLMW and Haar wavelets from [11], LLMW performs a better results in approximating the examples as shown in the tables (1) to (4). From the tables, it is clearly observed that the error estimation by linear Legendre multi-wavelets gives less error for the approximation. Therefore, it has been proved that the present method is more efficient and accurate than the Haar wavelets method.

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