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Variable order variable stepsize algorithm for solving nonlinear Duffing oscillator

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Abstract. Nonlinear phenomena in science and engineering such as a periodically forced oscillator with nonlinear elasticity are often modeled by the Duffing oscillator (Duffing equation). The Duffing oscillator is a type of nonlinear higher order differential equation. In this research, a numerical approximation for solving the Duffing oscillator directly is introduced using a variable order stepsize (VOS) algorithm coupled with a backward difference formulation. By selecting the appropriate restrictions, the VOS algorithm provides a cost efficient computational code without affecting its accuracy. Numerical results have demonstrated the advantages of a variable order stepsize algorithm over conventional methods in terms of total steps and accuracy.

1. Introduction

Many real life phenomena are modeled by systems of nonlinear higher order ordinary differential equations. These phenomenas are found in various fields such as engineering, physics, finance and communication theory with applications ranging from electrical circuits, high energy particles, to modern telecommunications. Due to its various applications, nonlinear higher order ordinary differential equations have been the subject of interest, particularly the Duffing oscillator.

The Duffing oscillator is a second order non-linear initial-value ordinary differential equation (ODE) with the general form

$$y''(t) + \delta y'(t) + \alpha y(t) + \beta y^3(t) = \gamma \sin \omega t, \quad (1)$$

where $\delta, \alpha, \beta, \gamma$ are constants and ω is a provided parameter.

Higher order ODEs such as the Duffing oscillator were previously reduced to a system of differential equations and solved using conventional numerical methods. In the current work, we propose to solve the Duffing differential equation directly using a variable order variable step multistep method.

Direct solutions for higher order ordinary differential equations have been popularized by authors such as [1–8] and etc. The techniques and strategies suggested by Suleiman in [9] is the foundation which many researches are based on.



A divided difference based Direct Integration (DI) method for solving higher order Odes directly was introduced by Suleiman in [4]. The DI method was then integrated with a 2-point explicit and implicit block difference formulation in [10]. In [6] a block backward differentiation formulae was derived for solving stiff Odes. Our current research focuses on a VOS algorithm in backward difference form for solving nonlinear Duffing differential equation.

The purpose of the backward difference formulation is to overcome the tedious calculation of integration coefficients at every step size change as required by a divided difference formulation. Because the integration coefficients of a backward difference formulation needs to be calculated only once, the computational cost is reduced. With an added advantage of a recurrence relationship between the integration coefficients for predictor and corrector, computational cost is reduced even more.

2. Variable order variable step size backward difference in predict-evaluate correct-evaluate (PeCe) mode

The formulation of a variable order stepsize algorithm predictor-corrector mode requires a set of integration coefficients and an order-step size strategy.

2.1. Derivation of the integration coefficients

Consider the d^{th} order ordinary differential equation

$$y^{(d)} = f(x, \tilde{Y}), \quad (2)$$

where $\tilde{Y}(\alpha) = \tilde{\eta}$ such that

$$\begin{aligned} \tilde{Y}(x) &= (y, y', \dots, y^{(d-1)}) \\ \tilde{\eta} &= (\eta, \eta', \dots, \eta^{(d-1)}) \end{aligned}$$

in the interval $\alpha \leq t \leq \beta$.

Integrating $y^{(d)}$, 1, 2, 3, ..., d number of times and interpolating $(y, y', \dots, y^{(d-1)})$ by the Newton-Gregory backward difference interpolation polynomial

$$P_{n+r}(x) = \sum_{i=0}^k (-1)^i \binom{-s}{i} \nabla^i f_{n+r}, \quad s = \frac{x - x_{n+r}}{h},$$

we have

$$y^{(d-j)}(x_{n+1}) = \sum_{i=0}^{j-1} \frac{(h)^i}{i!} y^{(d-j+i)}(x_n) + \int_{x_n}^{x_{n+1}} \frac{(x_{n+1} - x)^{d-1}}{(d-1)!} \sum_{i=0}^k (-1)^i \binom{-s}{i} \nabla^i f_{n+r} dx, \quad (3)$$

where $r = 0, 1$, and $j = 0, 1, \dots, d$ with the k^{th} order of backvalues.

When $r = 0$, the operator $\nabla^i f_{n+r}$ is used to obtain the predictor formula and $r = 1$ to obtain the corrector formula. Thus, the VOS algorithm is denoted in the following predictor-corrector mode:

Predictor:

$$y^{(d-j)}(x_{n+1}) = \sum_{i=0}^{j-1} \frac{h^i}{i!} y^{(d-j+i)}(x_n) + h^j \sum_{i=0}^{k-1} \gamma_{1,j,i} \nabla^i f_n$$

Corrector:

$$y^{(d-j)}(x_{n+1}) = \sum_{i=0}^{j-1} \frac{h^i}{i!} y^{(d-j+i)}(x_n) + h^j \sum_{i=0}^{k-1} \gamma_{1,j,i}^* \nabla^i f_{n+1}$$

Derivation of the integration coefficients can be obtained from [11]. This provide the following coefficients as shown below

Explicit Coefficients:

$$\gamma_{1,d,0} = \gamma_{1,d-1,1}, \quad \gamma_{1,d,k} = \gamma_{1,d-1,k+1} - \sum_{i=0}^{k-1} \left(\frac{\gamma_{1,d,i}}{k-i+1} \right), \quad k = 1, 2, \dots$$

Implicit Coefficients:

$$\gamma_{1,d,0}^* = \gamma_{1,d-1,1}^*, \quad \gamma_{1,d,k}^* = \gamma_{1,d-1,k+1}^* - \sum_{i=0}^{k-1} \left(\frac{\gamma_{1,d,i}^*}{k-i+1} \right), \quad k = 1, 2, \dots$$

In the next section we provide the order and step size strategy.

3. Order and step size

Integration step selection is essential in a variable order step size algorithm. The efficiency of the algorithm is dependent on the order and step size strategy selected whereas, its reliability is determined by the acceptance criteria. Order and step size strategy and acceptance criteria is crucial because implementing variable order in a multistep method relies on the back values stored. The order can be increased, depending whether the back values from the previous step have been discarded and decreased by discarding the suitable amount of back values. Results in [4] suggests an unbiased order strategy that is implemented in an Adams code are efficient for nonstiff problems. For purpose of the current work, the order strategy of choice is adopted from [12].

Let h be the calculated step size and h_{end} the final step size. A safety factor of R is multiplied with h such that $h_{end} = Rh$ for a conventional estimate of h_{end} and also to reduce number of steps rejected. Due to convergence and stability issues of variable step size techniques, Shampine and Gordon [12] recommends restrictions on ratio of successive step size to ensure stability. When applying a predictor-corrector (PECE) mode in backwards difference form, integrating the doubling or halving the step size algorithm (refer [13]) with a step size changing technique from [14].

Algorithm 1 Doubling step size algorithm

```

1:  $H_{min} := 0.8H_{min}$ 
2: If ( $H_{min} \geq 2$ )
3:    $H := 2H_{old}$ ;
4: If ( $H := 2H_{old}$ )
5:   For  $I := 1$ , to  $N$  step 1
6:     Begin
7:        $\nabla^{k-1} f_{n+1} := \frac{1}{2} \nabla^{k-1} f_{n+1}$ 
8:       For  $T := 1$ , to  $K - 2$  step 1
9:         Begin
10:        For  $M := T$ , to  $K - 2$  step 1
11:          Begin
12:             $\nabla^m f_{n+1} := 2(\nabla^m f_{n+1} - \nabla^{m+1} f_{n+1})$ 
13:          End
14:           $\nabla^{k-1} f_{n+1} := 2\nabla^{k-1} f_{n+1}$ 
15:        End
16:       $\nabla^{k-1} f_{n+1} := 2\nabla^{k-1} f_{n+1}$ 
17:    End

```

Algorithm 2 Halving step size algorithm

```

1:  $Errors := Errors + 1$ 
2:  $H := 0.5H_{old}$ 
3:  $X := X_{old}$ 
4: For  $I := 1$ , to  $N$  step 1
5:   Begin
6:      $\nabla^{k-1} f_{n+1} := \frac{1}{2} \nabla^{k-1} f_{n+1}$ 
7:     For  $T := K - 2$ , to 1 step -1
8:       Begin
9:          $\nabla^{k-1} f_{n+1} := \frac{1}{2} \nabla^{k-1} f_{n+1}$ 
10:        For  $M := K - 2$ , to  $T$  step -1
11:          Begin
12:             $\nabla^m f_{n+1} := \frac{1}{2} (\nabla^m f_{n+1} + \nabla^{m+1} f_{n+1})$ 
13:          End
14:        End
15:   End

```

Finally, we obtain the error estimation which is also the criteria for order and step size selection.

4. Error estimation

Estimation for the local errors of each integration step (see [3]) begins by denoting the predictor as follows

$$pr y_{n+1}^{(d-j)} = \sum_{i=0}^{j-1} \frac{h^i}{i!} y_n^{(d-j+i)} + h^j \sum_{i=0}^{k-1} \gamma_{1,j,i} \nabla^i f_n, \tag{4}$$

where the coefficients $\gamma_{d,i}$ is independent of k . When applying a $P_k EC_{k+1} E$ algorithm, the corrector has the form

$$cr y_{n+1}^{(d-j)} = \sum_{i=0}^{j-1} \frac{h^i}{i!} y_n^{(d-j+i)} + h^j \sum_{i=0}^{k-1} \gamma_{1,j,i}^* \nabla^i f_{n+1} \tag{5}$$

with the i -th backward difference of the predictor, ∇_{pr}^i using $f(x_{n+1}, \tilde{Y}_{n+1}^{pr})$ for f_{n+1} . For computational purposes and from (4), the corrector can be written as

$$cr y_{n+1}^{(d-j)} = pr y_{n+1}^{(d-j)} + \gamma_{1,j,i}^* \nabla_{pr}^i f_{n+1}$$

where $j = 0, 1, \dots, d$ and $i = 0, 1, \dots, k$.

The local truncation error (LTE) (by Milne error estimate), has the following formulation

$$\tilde{E}_{n+1,k}^{(j)} = h^j \gamma_{1,j,i}^* \nabla_{pr}^k f_{n+1}.$$

The selection of a suitable p for $\tilde{E}_k^{(d-p)}$ resembling the proof in [15] is to control order and step size. The asymptotic validity can be established using

$$\tilde{E}_{n+1,k+1}^{(d-p)} = h^{d-p} \gamma_{1,d-p,k+1}^* \nabla^{k+1} f_{n+1}.$$

The error analysis of the method is similar to [9].

5. Numerical results

For numerical solution of the Duffing oscillator, refer to the works such as [16–19]. The VOSBD method was tested on numerous nonlinear Duffing oscillators of different parameters. Problems 1-2 consist of non homogeneous Duffing oscillator. Problem 3 is a homogenous Duffing oscillator without an exact solution where as, Problem 4 is higher order (4th order) Duffing oscillator. We then evaluate the maximum and average error in the computed solution. The numerical result are then compared with conventional methods to validate its accuracy and with another variable order stepsize method, the DI method in particular to show its efficiency. The following notations are used

- | | |
|----------------------------------|---------------------------------------|
| STEPS: total steps, | DI: direct integration, |
| MAXE: the overall maximum error, | VOSBD: VOS backward difference, |
| AVER: the average error, | SHPM: standard homotopy perturbation, |
| MTD: the method used, | SNM: standard numerical. |

Table 1. Test problems for the Duffing oscillator.

No.	Problem	Condition	Exact Solution
1.	$y''(x) + 2y'(x) + y(x) + 8y^3(x) = e^{-3t}$, $0 \leq x \leq 100$, Source: [20].	$y(0) = \frac{1}{2}$, $y'(0) = -\frac{1}{2}$,	$y(x) = \frac{1}{2}e^{-t}$.
2.	$y''(x) + y(x) + y'(x) + y^2(x)y'(x) = 2 \cos x - \cos^3 x$, $0 \leq x \leq 100$, Source: [21].	$y(0) = 0$, $y'(0) = 1$,	$y(x) = \sin x$.
3.	$y''(x) + y(x) + y^3(x) = 0$, $0 \leq x \leq 5$, Source: [22].	$y(0) = 1$, $y'(0) = 0$,	
4.	$y''''(x) + 5y''(x) + 4y(x) - \frac{1}{6}y^3(x) = 0$, $0 \leq x \leq 1$, Source: [23].	$y(0) = 0$, $y'(0) = 1.91103$, $y''(0) = 0$, $y'''(0) = -1.15874$,	$y(x) = 2.1906 \sin 0.9x$ $-0.02247 \sin 2.7x$ $+0.000045 \sin 4.5x$.

Table 2. Comparison of total steps and accuracy for Problems 1 and 2.

TOL	MTD	Problem 1			Problem 2		
		STEPS	MAXE	AVER	STEPS	MAXE	AVER
10 ⁻²	DI	156	4.54154(-2)	1.12922(-2)	254	8.49079(-2)	2.04794(-2)
	VOSBD	154	5.54428(-2)	9.04569(-3)	217	1.07600(-1)	3.03894(-2)
10 ⁻⁴	DI	169	3.28968(-4)	8.48859(-5)	332	1.54704(-3)	4.72630(-4)
	VOSBD	215	4.90227(-4)	6.47465(-5)	284	1.24649(-3)	1.92899(-4)
10 ⁻⁶	DI	173	2.19533(-5)	2.80453(-6)	382	4.24089(-5)	1.56849(-5)
	VOSBD	236	1.31698(-5)	1.91707(-6)	330	1.28039(-5)	3.26641(-6)
10 ⁻⁸	DI	204	1.48896(-7)	3.10007(-8)	651	7.93605(-7)	1.55130(-7)
	VOSBD	224	1.82416(-7)	2.22705(-8)	499	7.27324(-7)	1.46368(-7)
10 ⁻¹⁰	DI	317	1.18565(-9)	3.94604(-10)	772	7.83863(-9)	2.03721(-9)
	VOSBD	224	1.12450(-9)	2.04848(-10)	702	9.05773(-9)	1.01381(-9)

Table 3. Numerical result for Problem 3.

x	VOSBD			SHPM	SNM
	$Tol = 1 \times 10^{-1}$	$Tol = 1 \times 10^{-5}$	$Tol = 1 \times 10^{-10}$		
0.5	7.68805(-1)	7.68804(-1)	7.68802(-1)	7.68766(-1)	7.68802(-1)
1.0	2.33744(-1)	2.33691(-1)	2.33692(-1)	2.33680(-1)	2.33692(-1)
2.0	-7.31500(-1)	-8.59353(-1)	-8.59349(-1)	-8.9323(-1)	-8.59349(-1)
3.5	-2.13813(-1)	-9.29568(-2)	-9.30110(-2)	-9.30340(-2)	-9.30130(-2)
5.0	5.83491(-1)	9.46863(-1)	9.47130(-1)	9.47107(-1)	9.47130(-1)

Table 4. Comparison of total steps and accuracy for Problem 4.

TOL	MTD	Problem 4		
		STEPS	MAXE	AVER
10^{-2}	DI	48	2.06305(-1)	3.75201(-2)
	VOSBD	46	2.19901(-2)	4.59352(-3)
10^{-4}	DI	80	6.74572(-4)	2.44601(-4)
	VOSBD	86	1.58280(-3)	3.94428(-4)
10^{-6}	DI	139	1.37722(-4)	4.02719(-5)
	VOSBD	102	1.26373(-4)	3.15140(-5)
10^{-8}	DI	399	1.28150(-4)	3.03717(-5)
	VOSBD	126	1.23095(-4)	2.94248(-5)
10^{-10}	DI	308	1.27015(-4)	3.42987(-5)
	VOSBD	225	1.19540(-4)	3.24947(-5)

6. Discussion and conclusion

Numerical results in Table 2 display the comparison of total steps and accuracy between the VOSBD and DI method for oscillating problems taken from [20] and [21]. Results of Problem 1 show the competitive nature between the VOSBD and DI method. The DI method shows to be better for larger tolerances whereas the numerical results favor the VOSBD method when using a more finer tolerance. In Problem 2, numerical results shows the efficiency of VOSBD over DI method in terms of both total steps and accuracy with the exemption of a couple of tolerances.

Table 3 entails comparison of the accuracy between the VOSBD and conventional methods for Problem 3 which was obtained from [22]. Due to the absence of the exact solution, the accuracy is compared with the analytic and numerical approximation provided in [22]. The approximation shown by the VOSBD method is for three different tolerance, 10^{-1} , 10^{-5} and 10^{-10} . From Table 3, it is clear that the numerical approximation by VOSBD method is valid especially when using finer tolerances.

The numerical results of VOSBD and DI method for Problem 4 (4^{th} order Duffing equation, see [23]) is provided in Table 4. The results of Table 4 clearly illustrate, even for higher order Duffing oscillators the VOSBD has the advantage in total steps compared to the DI method without lost of accuracy. The least number of steps correlates with computational cost (time).

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