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Numerical Integration Based on Linear Legendre Multi Wavelets

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Abstract. In the present work, a new direct computational method for solving definite integrals based on linear Legendre multi wavelets is introduced. This approach is an improvement of previous methods which are based on Haar wavelets functions. An algorithm using properties of the linear Legendre multi wavelets is developed in order to find numerical approximations for double, triple and improper integrals. The main advantage of this method is its efficiency and simple applicability. To validate the algorithm, numerical experiments are conducted to illustrate the accuracy of the method.

1. Introduction

In recent years, interest in the numerical solution of integrals has increased enormously. This is due to the fact that many applications in mathematics, physics and engineering require numerical integration. There is a number of related works devoted to the approximations involving quadrature rule for numerical integration. The classical approach in numerical integration is either Gaussian quadrature or Newton-Cotes formula [1, 2, 3, 4, 5, 6]. However, the numerical quadrature bears some drawbacks [7]. In the case of Newton-Cotes formula, the use of large number of equally spaced nodes may cause erratic behavior when implementing high degree polynomial interpolation. This is because a large number of node points are needed to get high accuracy in numerical integration. The Gaussian quadrature rule can be derived by the method of undetermined coefficients. Because the resulting equations for the $2n$ unknown nodes and weights are nonlinear which are not easy to solve. In order to overcome the obstacles, a new method based on wavelets approximation is propose to find the numerical solutions of integrals.

The current research improves the accuracy of works completed in [8]. Siraj-ul-Islam in [8], details the Haar wavelets base function used to solve definite multiple integral such as single, double, triple and improper integral. The Haar wavelets base function was embedded with a simple and direct algorithm for solving the integrals. There are



various types of wavelets with different properties, such as Haar wavelets, Daubechies' orthonormal of compact support and Chebyshev wavelets [9, 10, 11, 12].

In this article we propose a new algorithms by linear Legendre multi wavelet functions. The organization of this paper is as follows. In section 2 and 3, linear Legendre multi wavelets and numerical integration using linear Legendre multi wavelets has been described. In section 4 numerical results are reported. Further discussion are drawn in section 5. We proceed with a brief introduction of the linear Legendre multi wavelets in the following section.

2. Linear Legendre multi wavelets (LLMW)

In this paper we deal with the problems of approximation of the integrals by LLMW. The wavelets, in general, are very important tools for analyzing the signals in engineering sciences and consist of a family of functions constructed from dilation and translation of a single function called the mother wavelet. Here we start from introducing the linear Legendre mother wavelets functions as $\psi^0(x)$, $\psi^1(x)$ and the linear Legendre multi wavelets $\psi_{k,n}^j(x)$ [13].

The formula is described as follow:

$$\psi^0(x) = \begin{cases} -\sqrt{3}(4x-1), & \left[a, \frac{a+b}{2} \right) \\ \sqrt{3}(4x-3), & \left[\frac{a+b}{2}, b \right) \end{cases}, \quad \psi^1(x) = \begin{cases} 6x-1, & \left[a, \frac{a+b}{2} \right) \\ 6x-5, & \left[\frac{a+b}{2}, b \right) \end{cases},$$

and with dilation and translation the linear Legendre multi wavelets is

$$\psi_{kn}^j(x) = \begin{cases} 2^{\frac{k}{2}} \psi^j \left(2^k \frac{x-a}{b-a} - n \right), & a + n \frac{(b-a)}{2^k} \leq x < a + (n+1) \frac{(b-a)}{2^k}, \\ 0, & \text{otherwise,} \end{cases}$$

where $n = 0, 1, \dots, 2^k - 1$, $k \in \mathbb{Z}_+$, and $j = 0, 1$ are defined on the interval $[a, b)$. On the other hand the scaling functions be denoted as $\phi_0(x) = 1$, $\phi_1(x) = \sqrt{3}(2x-1)$, $a \leq x < b$. Any function $f(x) \in L^2(\mathbb{R})$ in the interval $[a, b)$ can be approximated as

$$f_M(x) = c_0 \phi_0(x) + c_1 \phi_1(x) + \sum_{k=0}^M \sum_{j=0}^1 \sum_{n=0}^{2^k-1} c_{kn}^j \psi_{kn}^j(x) = C^T \Psi(x), \quad (1)$$

where C and $\Psi(x)$ are matrices given by

$$C = [c_0, c_1, c_{00}^0, c_{00}^1, \dots, c_{M0}^0, c_{M1}^0, \dots, c_{M(2^M-1)}^0, c_{M0}^1, c_{M1}^1, \dots, c_{M(2^M-1)}^1]^T,$$

and

$$\Psi = [\phi_0, \phi_1, \psi_{00}^0, \psi_{00}^1, \dots, \psi_{M0}^0, \psi_{M1}^0, \dots, \psi_{M(2^M-1)}^0, \psi_{M0}^1, \psi_{M1}^1, \dots, \psi_{M(2^M-1)}^1]^T.$$

3. Numerical integration by LLMW

Consider the definite integral as follow:

$$\int_a^b f(x) dx.$$

The function $f(x) \in L^2(\mathbb{R})$ can be approximation by the expansion of Eq. (1) as

$$f(x) \approx C^T \Psi(x), \quad (2)$$

where $M \in \mathbb{Z}_+$ is the maximum $k = 0, \dots, M$ dilation for LLMV.

Lemma 1. *The approximate value of the integral is given by*

$$\int_a^b f(x) dx \approx c_0(b-a).$$

Proof: Since

$$\int_a^b \psi_{kn}^j(x) dx = 0, \quad \forall k = 0, 1, \dots, M, \quad n = 0, 1, \dots, 2^k - 1$$

and the scaling functions

$$\int_a^b \phi_1(x) dx = 0,$$

and

$$\int_a^b \phi_0(x) dx = 1,$$

therefore,

$$\begin{aligned} \int_a^b f(x) dx &\approx \int_a^b C^T \Psi(x) dx = c_0 \int_a^b \phi_0(x) dx + c_1 \int_a^b \phi_1(x) dx + \\ &\quad \sum_{k=0}^M \sum_{j=0}^{2^k-1} c_{kn}^j \int_a^b \psi_{kn}^j(x) dx = c_0(b-a). \end{aligned}$$

From above the valuation of the definite integral by LLMW if involves only one coefficient c_0 and usually to compute this coefficient c_0 we consider the nodal points

$$x_i = a + (b-a) \frac{i+0.5}{2^{k+2}}, \quad i = 0, 1, \dots, 2^{k+2} - 1.$$

Therefore discretized form of (2) can be written as

$$f(x_i) \approx c_0 \phi_0(x_i) + c_1 \phi_1(x_i) + \sum_{k=0}^M \sum_{j=0}^{2^k-1} c_{kn}^j \psi_{kn}^j(x_i), \quad (3)$$

and reduces the equation to a linear algebra equation, to obtain c_0 . However, this process takes too much calculation, just to find c_0 if we consider M to be a large value. A new approach in finding c_0 have been developed to reduces the computation effort.

Lemma 2. *The approximate value of the integral is given by*

$$c_0 = \frac{1}{2^{k+2}} \sum_{i=0}^{2^{k+2}-1} f(x_i).$$

Proof: By taking the sum of the left and right side by i from 0 to $2^{k+2} - 1$ gives us an easy formula to calculate the value of the coefficient c_0 :

$$\begin{aligned} \sum_{i=0}^{2^{k+2}-1} f(x_i) &= c_0 \sum_{i=0}^{2^{k+2}-1} \phi_0(x_i) + c_1 \sum_{i=0}^{2^{k+2}-1} \phi_1(x_i) + c_{00}^0 \sum_{i=0}^{2^{k+2}-1} \psi_{00}^0(x_i) + c_{00}^1 \sum_{i=0}^{2^{k+2}-1} \psi_{00}^1(x) + \dots \\ &\quad + c_{M0}^0 \sum_{i=0}^{2^{k+2}-1} \psi_{M0}^0(x_i) + c_{M1}^0 \sum_{i=0}^{2^{k+2}-1} \psi_{M1}^0(x_i) + \dots + c_{M(2^M-1)}^0 \sum_{i=0}^{2^{k+2}-1} \psi_{M(2^M-1)}^0(x_i) \\ &\quad + c_{M0}^1 \sum_{i=0}^{2^{k+2}-1} \psi_{M0}^1(x_i) + c_{M1}^1 \sum_{i=0}^{2^{k+2}-1} \psi_{M1}^1(x_i) + \dots + c_{M(2^M-1)}^1 \sum_{i=0}^{2^{k+2}-1} \psi_{M(2^M-1)}^1(x_i). \end{aligned}$$

From the above, all summations

$$\sum_{i=0}^{2^{k+2}-1} \left(\sum_{k=0}^M \sum_{j=0}^1 \sum_{n=0}^{2^k-1} \psi_{kn}^j(x_i) \right) = 0, \quad k = 0, 1, \dots, M, \quad n = 0, 1, \dots, 2^k - 1,$$

and

$$\begin{cases} \sum_{i=0}^{2^{k+2}-1} \phi_1(x_i) = 0, \\ \sum_{i=0}^{2^{k+2}-1} \phi_0(x_i) = 2^{k+2}, \end{cases}$$

for all $i = 0, 1, \dots, 2^{k+2} - 1$, therefor the lemma is true for c_0 and the solution of the system (3) has the following form:

$$c_0 = \frac{1}{2^{k+2}} \sum_{i=0}^{2^{k+2}-1} f(x_i).$$

Applying the ideas above we establish the following formula for numerical integral equation

$$\int_a^b f(x) dx \approx \frac{1}{2^{k+2}} \sum_{i=0}^{2^{k+2}-1} f(x_i) = \frac{(b-a)}{2^{k+2}} \sum_{i=0}^{2^{k+2}-1} f\left(a + (b-a) \frac{i+0.5}{2^{k+2}}\right). \quad (4)$$

By following the same procedures as implied above the LLMW-base algorithm can be easily extended to double integral

$$\int_a^b \int_a^b f(x, y) dx dy.$$

To illustrate the function $f(x, y)$ in term LLMW fuctions denoted

$$f(x, y) \approx \Psi^T(x) K \Psi(y), \quad (5)$$

where

$$K = \begin{bmatrix} c_0 c_0 & c_0 c_1 & \cdot & \cdot & \cdot & c_0 c_{M(2^M-1)}^1 \\ c_1 c_0 & c_1 c_1 & \cdot & \cdot & \cdot & c_1 c_{M(2^M-1)}^1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{M(2^M-1)}^1 c_0 & c_{M(2^M-1)}^1 c_1 & \cdot & \cdot & \cdot & c_{M(2^M-1)}^1 c_{M(2^M-1)}^1 \end{bmatrix}$$

Lemma 3. *The approximate value of the integral is*

$$\int_c^d \int_a^b f(x, y) dx dy \approx \Psi^T(x) K \Psi(y) = c_0 c_0 (b-a)(d-c).$$

Proof: The proof of this lemma is similar to the proof of lemma 2.1.

The process of finding $c_0 c_0$ is the same step apply in one dimensional case. Define the points

$$x_i = a + (b - a) \frac{i + 0.5}{2^{k+2}}, \quad i = 0, 1, \dots, 2^{k+2} - 1.$$

$$y_j = c + (d - c) \frac{j + 0.5}{2^{k+2}}, \quad j = 0, 1, \dots, 2^{k+2} - 1,$$

and substitud into (5) to obtain

$$f(x_i, y_j) \approx \Psi^T(x_i)C\Psi(y_j). \quad (6)$$

Lemma 4.

$$c_0c_0 = \frac{1}{2^{2k+4}} \sum_{i=0}^{2^{k+2}-1} \sum_{j=0}^{2^{k+2}-1} f(x_i, y_j).$$

Proof: The proof of this lemma is similar to the proof of lemma 2.2.

Hence the final formula for approximating the double integral by LLMW is

$$\int_c^d \int_a^b f(x, y) dx dy \approx \frac{(d - c)(b - a)}{2^{2k+4}} \sum_{i=0}^{2^{k+2}-1} \sum_{j=0}^{2^{k+2}-1} f(x_i, y_j),$$

and this formula can be extended to triple integral as

$$\int_e^f \int_c^d \int_a^b f(x, y, z) dx dy dz \approx \frac{(f - e)(d - c)(b - a)}{2^{3k+6}} \sum_{i=0}^{2^{k+2}-1} \sum_{j=0}^{2^{k+2}-1} \sum_{l=0}^{2^{k+2}-1} f(x_i, y_j, z_l).$$

where

$$x_i = a + (b - a) \frac{i + 0.5}{2^{k+2}}, \quad i = 0, 1, \dots, 2^{k+2} - 1,$$

$$y_j = c + (d - c) \frac{j + 0.5}{2^{k+2}}, \quad j = 0, 1, \dots, 2^{k+2} - 1,$$

$$z_l = e + (f - e) \frac{l + 0.5}{2^{k+2}}, \quad l = 0, 1, \dots, 2^{k+2} - 1.$$

4. Numerical Examples

In this section we wish to show the efficiency of LLMW by comparing it to the previous works [8]. They applied Haar wavelets to approximate the integral All the example has been solved numerically by LLMV. Numerical results for Haar and LLMW with the same level dilation, J is the dilation for Haar wavelets and maximum $k = M$ is the dilation for LLMW to ensure more accurate between both methods.

4.1. Test Problems

Example	Problem	Exact Solution
1.	$\int_0^1 \sin(x^2) dx$	
2.	$\int_0^5 \sqrt{x^2 - 5x + 31} dx$	$\frac{5\sqrt{31}}{2} + \frac{99}{4} \sinh^{-1} \left(\frac{5}{3\sqrt{11}} \right)$
3.	$\int_0^1 \frac{e^{-\frac{1}{x}}}{x^2} dx.$	e^{-1}
4.	$\int_0^{\frac{\pi}{2}} \int_0^{\pi} \sin(x + y) dx dy$	2
5.	$\int_0^1 \int_0^1 \frac{1}{\sqrt{x^2+y^2}} dx dy$	$\ln(1 + \sqrt{2}) + \tanh^{-1} \left(\frac{\sqrt{2}}{2} \right)$
6.	$\int_1^2 \int_1^2 \int_1^2 \frac{1}{x+y+z} dx dy dz$	$4 \log \left(\frac{11943936}{9765625} \right) + \log \left(\frac{2500\sqrt{\frac{5}{3}}}{59049} \right)$

4.2. Numerical Results

LLMW	Rel. errors	Haar	Rel. errors
$k = 4$	3.5431E-05	$J = 4$	1.4177E-04
$k = 5$	8.8562E-06	$J = 5$	3.5432E-05
$k = 6$	2.2122E-06	$J = 6$	8.8574E-06
$k = 7$	5.5307E-07	$J = 7$	2.2143E-06

Table 1: Relative error for Example 1

LLMW	Rel. errors	Haar	Rel. errors
$k = 4$	8.8228E-06	$J = 4$	3.5293E-05
$k = 5$	2.2063E-06	$J = 5$	8.8229E-06
$k = 6$	5.5321E-07	$J = 6$	2.2057E-06
$k = 7$	1.3946E-07	$J = 7$	5.5143E-07

Table 2: Relative error for Example 2

LLMW	Rel. errors	Haar	Rel. errors
$k = 4$	1.0172E-05	$J = 4$	4.0642E-05
$k = 5$	2.5427E-06	$J = 5$	1.0173E-05
$k = 6$	6.3526E-07	$J = 6$	2.5431E-06
$k = 7$	1.5602E-07	$J = 7$	6.3578E-07

Table 3: Relative error for Example 3

LLMW	Rel. errors	Haar	Rel. errors
$k = 4$	1.2550E-04	$J = 4$	5.2015E-04
$k = 5$	3.1356E-05	$J = 5$	1.2551E-04
$k = 6$	7.8390E-06	$J = 6$	3.1375E-05
$k = 7$	1.9275E-06	$J = 7$	7.8437E-06

Table 4: Relative error for Example 4

LLMW	Rel. errors	Haar	Rel. errors
$k = 4$	3.5718E-03	$J = 4$	7.1275E-03
$k = 5$	1.7880E-03	$J = 5$	3.5719E-03
$k = 6$	8.9449E-04	$J = 6$	1.7880E-03
$k = 7$	4.4740E-04	$J = 7$	8.9450E-04

Table 5: Relative error for Example 5

LLMW	Rel. errors	Haar	Rel.e errors
$k = 4$	3.2103E-06	$J = 4$	1.2863E-05
$k = 5$	1.7970E-07	$J = 5$	3.2159E-06
$k = 6$	1.9506E-07	$J = 6$	8.0398E-07

Table 6: Relative error for Example 6

5. Discussion

A comparative analysis between Haar and LLMW functions in terms of relative error is performed to find numerical approximations of different types of integral. In Tables (1-6), we compared the numerical results obtained by LLMW with Haar wavelets from [8]. Comparison between the two methods show the obvious advantage of LLMW method over the Haar wavelets method.

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