CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

The theory of IEs has been studied since the early nineteenth century and has close relations with many different areas of science. Many problems of mathematical physics can be formulated in the form of IEs.

The first classical theory of IEs was by Fredholm, who used the potential for troubleshooting and in elastostatic boundary value. Many problems in elastic deformation; the boundary value problems in mathematical physics and engineering are recast to SIEs of the first or second kind (Becker (1992), Dzhuraev (1992) and Kanwal (2013)). These different problems have led researchers to establish different methods for solving the IEs analytically and numerically as cited by Golberg (1979) and Gakhov (1963), where the authors solved the SIEs for those interested in applications to the theory of elasticity and hydromechanics. Many authors, like Porter & Stirling (1990) and Golberg (2013), implemented the CSIEs. However, these applications are limited to certain situations.

For the first time, the numerical solution of the IE appears in many papers; thus, many researchers began to take interest in this area, covering the most diverse areas of engineering such as physical, geometric, plasticity, propagating voltages, viscoelasticity, in the tension; mechanics of soil and rocks, contact, percolation, fracture mechanics, vibrations, wave propagation, soil-structure interaction, plates, shells, stress concentration, combinations with other methods for the analysis of various problems, among others Gakhov (1963) and Kravchenko & Litvinchuk (2012).

In recent years, much attention has been driven to the numerical methods not only on applied mathematics but also in the general sciences. In addition, many articles on numerical analysis for the IEs Golberg (1979) and Golberg (1984a), contained applications of SIEs. The computational implementations were carried out, and a numerical approach was presented to solve the problem. Researchers identified that individual cores appearing in potential problems and the whole that existed towards the Cauchy principal value (CPV) should have a special attention.

In the last four decades or more, many authors have described, discussed and studied the numerical solution of CSIEs and among the methods, in these studies the direct quadrature method are used, see Tuck (1980). Jen & Srivastav (1981) applied the spline approximation method to obtain a numerical solution of CSIEs; the density function is expanded as the product of an appropriate weight function and a cubic spline, which can reduce the problem to a system of linear algebraic equations.

From these studies, the scientific communities aroused curiosity of CSIEs. The method had a generalization to engineering problems with the work, in which solving techniques of IEs were calculated numerically using different methods, as collocation, Galerkins and Nystrom methods (Golberg (1984a), Noble (1973) and Assari et al. (2014)). From there the SIEs begin to be seen as numerical methods.

One of the formulations which singular terms used in the sense CPV was developed by Li & Chen (2007) who established Gauss quadrature rules for solving CSIEs and constructed orthogonal polynomials of scaling functions and wavelets as weight functions.

In general, projection method plays a great significance role in numerical analysis. In particular, in the Galarkin, the collocation methods by Ioakimidis (1981) and Phillips (1972) have importan applications in different types of problems of the numerical mathematics such as IEs with singular kernel, they are an effective means of numerically solving CSIEs. Mennouni & Guedjiba (2011) presented a projection method for solving bounded operators in Hilbert spaces and applied the method for solving the CSIEs by using a sequence of orthogonal function with finite rank projections.

Moreover, recent work has shown a wide applications of Galerkin method and Galerkin analysis methods by using piecewise polynomials. In particular, much work has been done on the effects of numerical integration on their convergence. Many authors have described and discussed the convergence of discrete Galerkin methods (Golberg et al., 1994). Assari et al. (2014) describes a new technique based on the discrete Galerkin to solve LSIEs of the second kind.

Gong (2007) proposed a method based on Galerkin method, and also provided the error bound and the rate of convergence for this method. Deloff (2008) presented an exact expressions for efficient computation of the weights in Gauss-Legendre and Chebyshev quadratures for selected SIEs included in CSIEs. More information about the numerical methods in various areas of applied mathematics can be found in Golberg (1979).

In these studies, the integrals are calculated using the CPV direction Hadamard finite portions and applies a system of orthogonal coordinates in the point located inside there by calculating terms which appear at regular integrals. The numerical solution found shows effectiveness when compared to the classic formulation, thus validating its formulation. More work by Muskhelishvili & Radok (2008) developed the theory of SIEs, they assumed a different technique and increased important applications in various areas of science. Since analytic solution to these of SIEs are usually not available, more attention has been concentrated on the numerical treatment cited by Golberg (1984b).

The CSIEs play an important role in many fields of science, and many different numerical approximations are obtained. Indeed Gerasoulis (1982) used a piecewise quadratic polynomials in the solution of the SIEs.

Traditionally in literature, the CSIEs are solved numerically by piecewise polynomial collocation methods. However, there are many cases where the approximation solutions are based on polynomials for IE in one dimension.

Berthold et al. (1992) discussed CSIEs with the error estimate under a Sobolev norm; they devised a fast algorithm for the quadrature method which permits the solution of the linear system arising from the method; moreover, the uniform convergence of this algorithm was presented in Junghanns & Kaiser (2013).

Linz (1977) expand the kernel of CSIEs in terms of Chebyshev polynomials. He establish convergence rates of an approximate solution. Moreover, Ioakimidis (1981) proposed a numerical method for the numerical solution of CSIEs based on the approximation of the density function by truncated series of orthogonal polynomials. He shows that the identical corresponding method depends on the reduction of CSIEs to an equivalent Fredholm IE of the second kind and the application to the latter of the weighted Galerkin method.

Gong (2007) gave a new error estimate under the weighted Chebyshev norm. Miller & Keer (1985) extended the piecewise quadratic method in terms of truncated series to solve SIEs of the first kind, Abdou & Nasr (2003) treated the SIEs based on Orthogonal polynomials method. The technique depends on isolating the singular part of the kernel, which corresponds to the selected domain of parameter variation and expanded the unknown and density functions in Chebyshev polynomials.

On the other hand, Gong (2009) used Galerkin methods to approximate the CSIEs of the second kind with constant coefficients; he decomposed the unknown function as a terms of Jacobi polynomials and used it in the discussions. More references related with this topic can be found in the book (Elliott, 1982). Venturino (1992) used Galerkin method to get error bounds for CSIEs. Ramachandran (1993) proposed numerical methods based on Gaussian quadrature to solve LSIEs of the first kind, product integration methods.

Bialecki & Stenger (1988) developed numerical method based on trigonometric functions (sine functions) to get an approximate solution of CSIEs. They obtained an approximate solution to the Fredholm IEs by means of Nystrom's method based on a smooth sine quadrature rule .

Considerable efforts have been devoted to achieving good accuracy for approximation CSIEs. Eshkuvatov et al. (2009) proposed efficient approximation methods to solve the first kind of CSIEs. They chose the zeros of four kinds of Chebyshev polynomials as collocation points. The numerical solutions found show effectiveness when compared to the classic methods. Abdou (2003) presented a numerical treatment to solve CSIEswhere the singular term has been eliminated and the approximate solution can be expanded in terms of Chebyshev polynomials form to get a system of linear algebraic equation. Okecha (2007) presented an efficient quadrature rule based on Lagrange interpolation and Gauss-Jacobi quadrature. The unknown function is expressed as a product of two functions; one of them is a regular function and other is the weight function related to Jacobi polynomials.

LSIEs are usually difficult to solve analytically. It needed to obtain an efficient approximate solution. The projection method by Atkinson (1997) is the commonly used approaches for the numerical solutions of these types of IEs. The piecewise polynomial collocation and Galerkin methods, Pedas & Vainikko (2006). Cao et al. (2007) studied the numerical solution of weakly singular IE by using the hybrid collocation methods; they proved the optimal order of convergence for this method.

Hasegawa & Sugiura (2007) presented an approximation solution based on quadrature method for LSIEs and approximated unknown function into finite sum of Chebyshev polynomials and expanded the given indefinite integral in terms of Chebyshev polynomials by using auxiliary algebraic logarithmic functions, by making a good approach to formulating CSIEs using polar coordinates.

Eshkuvatov et al. (2010) developed new quadrature formulas based on the modification of discrete vortex method for approximation CSIEs and presented results that confirmed the effectiveness of the method with discussion of convergence and error bounds. Sizikov & Sidorov (2016) proposed the generalized QMs for IE. They reduced it to nonsingular system of linear algebraic equations; error bounds are discussed and derived.

The numerical treatment of SIEs has been discussed by many researchers and many numerical methods have been proposed. Okecha & Onwukwe (2012) developed two efficient quadrature formulae for evaluating numerically SIEs. The authors applied four special cases of the zeros Jacobi polynomials.

Bougoffa et al. (2013) applied the ADM to get a new approach for resolving CSIEs by transforming it into a suitable form. Setia (2014) developed a numerical method based on Bernstein polynomials for solving various cases of CSIEs.

Abdulkawi (2015) extended the differential transform method for solving the CSIEs with two kernels types degenerate and convolution over a finite interval. In Araghi & Noeiaghdam (2016b), the authors presented a new effective and applicable method for solving CSIEs of the first kind and transform the first kind of CSIEs into second kind, then applied the homotopy analysis method to solve the obtained IE. Dezhbord et al. (2016) investigated the numerical solution of CSIEs based on reproducing kernel Hilbert space method.

2.2 Background

In this section, we discuss briefly some mathematical background, which is used directly to describe IEs. We need some mathematical concepts to carry the algebraic structure of functions and the operations that we perform on them. The main subject is that of linearity, a concept that prevailS, the theory of integral equations and functional analysis in general.

Many physical problems are modeled by IEs. The numerical solutions of IEs have been highly studied by many authors. In recent years, numerous studies have focused on the development of more advanced and efficient, effective methods for IEs such as the Bessel polynomial and the collocation approach.

Numerical solutions substantially reduced to systems of linear algebraic equations.

More importantly linearity is obvious in the functions and equations themselves concepts that apply in infite dimensional function spaces.

We introduce a brief of IEs and some of the main characteristic occur repeatedly in various styles. In the study of IEs from least to most specific, these spaces are the vector space, normed linear space, Banach space and Hilbert space. The material in this section is only apparent overview, for a much greater depth see any introductory texts on real analysis, for examples see Eidelman et al. (2004), Zarowski (2004) and Boyd (2001).

IEs occur in many practical applications, however, very few cases that can be solved analytically. For all other cases, they must be solved numerically. An integral equation is an equation with unknown function $\phi(x)$, arise within an integral sign. A typical of an IE in $\phi(t)$ is the form:

$$\phi(x) = g(x) + \int_{\alpha(x)}^{\beta(x)} k(x,t)\phi(t)dt, \tag{2.1}$$

where $k(x,t)$ is called the kernel of the IE, $\alpha(x)$ and $\beta(x)$ are the limits of integration. An IEs are called linear IEs when the unknown function $\phi(t)$ under the integral sign occur linearly, otherwise it was called nonlinear.

The most frequently used linear IEs are divided under two main classes namely Fredholm and Volterra IEs. However, the classification of IEs focuses on three basic categories which together describe the structure of linear IEs given as follows:

1. **Fredholm IEs:**
   The standard form of Fredholm linear IEs are given by

$$\phi(x)u(x) = g(x) + \lambda \int_a^b k(x,t)\phi(t)dt, \qquad x,t \in [a,b], \tag{2.2}$$

   where a, b are constants, $g(x)$ is given.
   The value of $u(x)$ will give the following kinds of Fredholm linear IEs:

   (a) when $u(x) = 0$, Eq. (2.2) becomes

$$g(x) + \lambda \int_a^b k(x,t)\phi(t)dt = 0, \qquad x,t \in [a,b] \tag{2.3}$$

   and is called Fredholm IEs of the first kind.

   (b) when $u(x) = 1$, Eq. (2.2) becomes

$$\phi(x) = g(x) + \lambda \int_a^b k(x,t)\phi(t)dt, \qquad x,t \in [a,b] \tag{2.4}$$

   and is called Fredholm IEs of the second kind.

2. **Volterra IEs:**
   The standard form of Volterra IEs are the form

$$\phi(x)u(x) = g(x) + \lambda \int_a^x k(x,t)\phi(t)dt \tag{2.5}$$

   Eq. (2.5) can be viewed as a special case of Fredholm IEs. The value of $u(x)$ will give the following kinds of Volterra linear IEs:

   (a) when $u(x) = 0$, Eq. (2.5) becomes:

$$g(x) + \lambda \int_a^x k(x,t)\phi(t)dt = 0, \tag{2.6}$$

   and is called Volterra IEs of the first kind.

   (b) when $u(x) = 1$, Eq. (2.5)becomes:

$$\phi(x) = g(x) + \lambda \int_a^b k(x,t)\phi(t)dt, \tag{2.7}$$

   and is called Volterra IEs of the second kind.

3. **Singular Integral Equations (SIEs):**
   The adjective singular depends on some time when the integration is improper, either the integral is infinite or the integral is unbounded with the given domain; may be an IE can be singular on both cases; i.e, at least one of the integration limits or interval is infinite or the kernel is unbounded in the domain, the adjective singular can be divided into two parts:

   (a) **Weakly** singular kernel:

$$k(x,t) = \frac{\hat{k}(x,t)}{|x-t|^{\alpha}}, \tag{2.8}$$

   where $\alpha \in (0,1)$ and $\hat{k}(x,t)$ is bounded. In particular the logarithmically singular kernel

$$k(x,t) = \hat{k}(x,t)\log|x-t|, \tag{2.9}$$

   where $\hat{k}(x,t)$ is bounded may be considered as weakly singular

   (b) **Strongly** singular kernel or Cauchy singular kernel:

$$k(x,t) = \frac{\hat{k}(x,t)}{x-t} \tag{2.10}$$

The homogeneity property: if we put $g(x) = 0$ in Fredholm or Volterra IEs of the second kind given by Eq. (2.2) and (2.5), the resulting equation is called a homogeneous IEs otherwise it is called inhomogeneous IEs. For more details see Wazwaz (2015) .

2.2.1 Eigenvalues and Eigenfunction

If we write the homogeneous IE as

$$\int_a^b k(x,t)\phi(t)dt = \mu\phi(x), \qquad \mu = \frac{1}{\lambda}, \qquad x,t \in [a,b], \tag{2.11}$$

then we have the classical eigenvalue problem; μ is the eigenvalue and $\phi(t)$ is the corresponding eigenfunction.

Definition 1 *(Kanwal, 2013) A function $\phi(t)$ defined on a set D is said to satisfy the Holder condition with exponent α if for any $t_1, t_2 \in D$, the inequality*

$$|\phi(t_1) - \phi(t_2)| \leq M|t_1 - t_2|^{\alpha} \tag{2.12}$$

holds with constant $M > 0$ and $0 < \alpha \leq 1$.

We simply say that the function $\phi(t)$ satisfies the H-condition or belongs to the class H on the set D. Such a function $\phi(t)$ is also called to be Holder continuous. We usually write $\phi(t) \in H(\alpha)$ or $\phi(t) \in H^{\alpha}(M,D)$.

2.2.2 Cauchy Principal Value (CPV)

Consider a function $\phi(t)$, defined in the interval $t \in [a,b]$, which is unbounded in the neighborhood of a point $d, a < d < b$, but is integrable in each of the intervals $(a, d-\varepsilon)$ and $(d+\eta, b)$ where ε and η are arbitrary, small positive numbers. Then, the limit

$$\int_a^b \phi(t)dt = \lim_{\varepsilon,\eta \to 0}\left[\int_a^{d-\varepsilon} \phi(t)dt + \int_{d+\eta}^b \phi(t)dt\right] \tag{2.13}$$

if it exists, is called the improper integral of the function $\phi(t)$ in the range (a,b). Here, it is implied that ε and η tend to zero independently. But it may happen that the limit does not exist when ε and η tend to zero independently of each other, but it exists if ε and η are related, then the above limit exists. In the special case $\varepsilon = \eta$, this limit is called the Cauchy principal value (CPV) or Cauchy principal integral Kanwal (2013).

2.3 Orthogonal Polynomials

Orthogonal polynomials play an important role in numerical analysis, mainly because functions belonging to general classes can be expanded in senes of orthogonal functions, such as, Fourier series and Fourier-Bessel series. An important class of orthogonal systems include orthogonal polynomials $p_n(x), (n = 0, 1, 2, \dots)$, where n is the degree of the polynomial $p_n(x)$. This class consists of many special functions commonly faced in the many applications, such as, Legendre, Jacobi, Hermite, Laguerre, and Chebyshev polynomials. Moreover, the orthogonality property of these polynomials have many other general properties, in which the contributions of researcher mathematicians like Bernstein and Chebyshev play a vital role (Boyd, 2001).

One of the most orthogonal polynomials that plays an important role in SIEs are the four kinds of Chebyshev polynomials. The following section is devoted to define Chebyshev polynomials with some theorems and basic properties (Mason & Handscomb, 2002).

2.3.1 The Chebyshev Polynomials

Definition 2 *Two functions $f(x)$ and $g(x)$ in $L_2[a,b]$ are said to be orthogonal on the interval $[a,b]$ with respect to a given continuous and non-negative weight function $w(x)$ if*

$$\int_a^b w(x) f(x) g(x) dx = 0, \tag{2.14}$$

If, for convenience, we use the inner product notation

$$\langle f, g \rangle_{w(x)} = \int_a^b w(x) f(x) g(x) dx \tag{2.15}$$

where $w(x), f(x)$ and $g(x)$ are functions of x on $[a,b]$, then the orthogonality condition (2.15) is equivalent to saying that $f(x)$ is orthogonal to $g(x)$ if

$$\langle f, g \rangle = 0 \tag{2.16}$$

The formal definition of an inner product real functions of a real variable is as follows.

Definition 3 *An inner product $\langle .,. \rangle$ is a linear function of elements $f, g, h, \dots$ of a vector space that satisfies the following axioms:*

1. $\langle f, f \rangle = 0$ with equality if and only if $f = 0$,
2. $\langle f, g \rangle = \langle g, f \rangle$,
3. $\langle f + g, h \rangle = \langle f, g \rangle + \langle f, h \rangle$,

4. $\langle \alpha f, g \rangle = \alpha \langle f, g \rangle$ for any scalar α.

An inner product defines an L_2 type norm

$$\|f\| = \|f\|_2 = \sqrt{\langle f, f \rangle}. \tag{2.17}$$

Definition 4 *(Mason & Handscomb, 2002) The Chebyshev polynomial $T_n(x)$ of the first kind is a polynomial in x of degree n, defined by the relation*

$$T_n(x) = \cos(n\theta), x = \cos(\theta), n = 0, 1, 2, \ldots \tag{2.18}$$

Definition 5 *The Chebyshev polynomials $U_n(x)$ of the second kind is a polynomial in x of degree n, defined by the relation*

$$U_n(x) = \frac{\sin((n+1)\theta)}{\sin\theta}, x = \cos(\theta), n = 0, 1, 2, \ldots \tag{2.19}$$

The range of the variables x and θ are the same as for $T_n(x)$.

Definition 6 *The Chebyshev polynomials of the third kind $V_n(x)$ is a polynomials in x of degree n, defined by the relations*

$$V_n(x) = \frac{\cos((n+\frac{1}{2})\theta)}{\cos\frac{1}{2}\theta}, \tag{2.20}$$

$n = 0, 1, 2, \ldots$ when $x = \cos(\theta)$.

Definition 7 *The Chebyshev polynomials of the fourth kind $W_n(x)$ is a polynomial in x of degree n, defined by the relations*

$$W_n(x) = \frac{\sin((n+\frac{1}{2})\theta)}{\sin\frac{1}{2}\theta}, when, x = \cos(\theta), \tag{2.21}$$

$n = 0, 1, 2, \ldots,$.

Theorem 2.3.1 *The Chebyshev polynomials $T_n(x), U_n(x), V_n(x), W_n(x)$ for $n = 0, 1, 2, \ldots$ are a set of orthogonal functions over $[-1, 1]$, with respect to the weight functions $w(x) = \sqrt{1-x^2}, \frac{1}{\sqrt{1-x^2}}, \sqrt{\frac{1-x}{1+x}}, \sqrt{\frac{1+x}{1-x}}$, respectively, $n, m = 0, 1, 2, 3, \ldots$, Moreover*

$$\langle T_n, T_m \rangle = \int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}} dx = \begin{cases} 0, & m \neq n, \\ \frac{\pi}{2}, & m = n, \\ \pi, & m = n = 0. \end{cases} \tag{2.22}$$

$$\langle U_n, U_m \rangle = \int_{-1}^{1} \sqrt{1-x^2} U_n(x) U_m(x) dx = \begin{cases} 0, & m \neq n, \\ \frac{\pi}{2}, & m = n, \\ \pi, & m = n = 0. \end{cases} \tag{2.23}$$

$$\langle V_n, V_m \rangle = \int_{-1}^{1} \frac{\sqrt{1+x}}{\sqrt{1-x}} V_n(x) V_m(x) dx = \begin{cases} 0, & m \neq n, \\ \pi, & m = n. \end{cases} \tag{2.24}$$

and

$$\langle W_n, W_m \rangle = \int_{-1}^{1} \frac{\sqrt{1-x}}{\sqrt{1+x}} W_n(x) W_m(x) dx = \begin{cases} 0, & m \neq n, \\ \pi, & m = n. \end{cases} \tag{2.25}$$

The proof of Theorem 2.3.1 and more detail can be found in Mason & Handscomb (2002) and Berthold et al. (1992).

Theorem 2.3.2 (Residue Theorem) *Let C be a closed path within and on which f is holomorphic except for m isolated singularities. Then*

$$\oint_C f(z)dz = 2\pi i \sum_{n=1}^{m} Res(f(z), z_n),$$

and the residue for z_0 pole of order n,

$$Res(f(z), z_n) = \lim_{x \to z_0} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left[(z - z_0)^n f(z) \right]. \tag{2.26}$$

The proof of Theorem 2.3.2 can be found in Spiegel (1965).

2.4 Projection Methods

Projection methods (PMs) have been around for a long time, and their general abstract treatment can be traced to the paper of Kantorovich in mid-twentieth century. Indeed Kantorovich & Akilov (1964) set general schema of defining and analyzing approximate methods to solve linear operator equations. The treatment given in this section is much simpler than that of it, we discuss briefly a special case of projection methods. More discussion and analyzing can be found in Atkinson (1997), Golberg (1979) and Elliott (1982).

Other discussions that have influenced a treatment are given in the papers of Prenter (1973), Phillips (1972) and Noble (1973). Elliott (1982) presented a paper that contains a collocation method based on the properties of Chebyshev polynomials and Chebyshev expansions, and this often leads to excellent convergence with a linear system of relatively small order.

2.4.1 Orthogonal Chebyshev Polynomials with Galerkin methods

The PMs, such as Galerkin method, are used mostly for the solution of all types of IEs, However, much of the analysis in the past four decades has involved operations, such as integration and functional space that are performed exactly, which is certainly not possible in virtually all realistic situations.

Practically, it becomes important to determine the effects of numerical integration, function approximate. Recently, the convergence analyses of Cauchy and logarithmic SIEs have taken up more attention and many researchers with related work have done for very general classes of IEs.

In this section, we briefly discuss and describe the approximation solution of linear operator considering their projections into finite-dimensional subspaces, which is more convenient for practical calculations. It is assumed that all operators are linear and bounded. Then, the general results concerning the convergence of the Galerkin methods will be applied to solving LogSIEk and SSIEk.

Definition 8 *Let X and Y be two Banach spaces and suppose $T : X \to Y$ be one-to-one operator. Assume that $X_n \subset X$ and $Y_n \subset Y$, be two sequences of subspaces with $dimX_n =$*

$dimY_n = n$ defined $P_n : Y \to Y_n$ be the projection operators The projection method generated by X_n and P_n approximates the equation

$$T\phi = f \tag{2.27}$$

by its projection

$$P_n T\phi = P_n f. \tag{2.28}$$

This PMs is called convergent for operator T, if there exists a natural number N such that for each $f \in ImT$, Eq. (2.28) has a unique solution $\phi_n \in X_n, \forall n \geq N$ these solutions converge $\phi_n \to \phi$ when $n \to \infty$ to the unique solution ϕ of $T\phi = f$, the convergence of the projection method in operators theory means that for all $n \geq N$. The finite-dimensional operators T_n; $P_n T : X_n \to Y_n$ are invertible and the pointwise convergence $T_n^{-1} P_n T\phi \to \phi$ when $n \to \infty$ holds $\forall \phi \in X$. In the general form, one may anticipate that the convergence happens if subspaces X_n form a dense set

$$\inf_{\varphi \in X_n} \|\varphi - \phi\| \to 0, n \to \infty, \forall \varphi \in X. \tag{2.29}$$

Formula (2.29) is called the approximating property, that is any element $\phi \in X$ can be approximated by elements $\varphi \in X_n$ with an arbitrary norm of X. Since $T_n = P_n T$ is a linear operator acting in finite-dimensional spaces, the PMs reduces to solve a finite-dimensional linear system. The Galerkin method will be considered as the PMs under the assumption that the above definition is valid in terms of the orthogonal projection. Indeed, for operator equations in Hilbert spaces, the PMs acting an orthogonal projection into finite-dimensional subspaces is called the **Galerkin method**.

2.4.2 Collocation Methods

Both Galerkin and Collocation Methods begin the same way. One seeks an approximation to y, the solution of Eq. (2.27) of the form $y_h \in S_h$ where S_h is a suitable finite dimensional linear space. The space S_h might be a space of piecewise polynomials. The two methods differ in the way in which the approximation y_h is selected from the space S_h. In the collocation method, one chooses a set of distinct points $t_1, t_2, t_3, \ldots, t_n \in [a,b]$ where $n = n_h$ is the dimension of S_h, and imposes the conditions

$$y_h(t_j) - ky_h(t_j) - f(t_j) = 0, i = 1, 2, \ldots, n \tag{2.30}$$

In practice, of course one chooses a basis $u_1, u_2, u_3, \cdots, u_n$ for S_h, so that y_h is a linear combination

$$y_h = \sum_{i=1}^{n} a_i u_i, \tag{2.31}$$

Then the coefficients $a_1, a_2, a_3, \ldots, a_n$ are determined in the collocation method by the set of linear equations

$$\sum_{i=1}^{n} a_i \Big[u_i(t_j) - k u_i(t_j) \Big] = f(t_j), j = 1, 2, \ldots, n. \tag{2.32}$$

Let the IE being solved be denoted by

$$T\phi = f \tag{2.33}$$

Indeed, both Galerkin and collocation methods approximate ϕ with functions from a finite-dimensional family, denoted by H_n. The methods try to minimize

$$R_n = T\phi_n - f \tag{2.34}$$

as ϕ_n ranges over H_n, in hope that the minimizer ϕ_n will also be close to ϕ. The methods differ in the criterion for making R_n small.

On the one hand, Galerkin methods minimize R_n by forcing the Fourier coefficients of R_n to be zero for all elements of some basis of X_n. If we assume that $\{\varphi_1, \varphi_2, \varphi_3, \ldots, \varphi_n\}$ is such a basis, then we choose $\phi_n \in H_n$ such that:

$$\langle R_n, \varphi_i \rangle = 0, \qquad i = 1, 2, 3, \ldots n, \tag{2.35}$$

where $\langle .,. \rangle$ denotes of inner product (see Eq. (2.15)). Writing

$$\phi_n = \sum_{j=1}^{n} a_i \varphi_j \tag{2.36}$$

the coeffiecients a_i are determined from

$$\sum_{j=1}^{n} a_i \langle T\varphi_j, \varphi_i \rangle = \langle f, \varphi_i \rangle, \qquad i = 1, 2, 3, \ldots n, \tag{2.37}$$

Hence in most cases, the inner products are integrals which must be evaluated numerically.

On the other hand, collocation methods make R_n small by forcing $R_n(t_i), i = 1, 2, 3, \ldots, n$ to be zero at a preselected set of points t_i in the integration domain $[a, b]$. Letting $\{t_1, t_2, \ldots, t_n\}$ be points in $[a, b]$, we need

$$R_n(t_i) = 0, \qquad i = 1, 2, 3, \ldots n, \tag{2.38}$$

If $\{\varphi_1, \varphi_2, \varphi_3, \ldots, \varphi_n\}$ is a basis of H_n, then we seek the solution ϕ_n in the form Eq. 2.36, the coeffiecients a_i are determined from

$$\sum_{j=1}^{n} a_i T\varphi_j(t_i) = f(t_i), \qquad i = 1, 2, 3, \ldots n, \tag{2.39}$$

2.4.3 Solution of LSIEs via Fourier–Chebyshev series

One can decompose functions in the Fourier series in Chebyshev polynomials (the Fourier–Chebyshev series)

$$g(x) = \frac{a_0}{2}T_0(x) + \sum_{i=1}^{\infty} a_i T_i(x), \qquad x \in [-1,1], \tag{2.40}$$

or

$$g(x) = \sum_{i=1}^{\infty} b_i U_i(x), \qquad x \in [-1,1], \tag{2.41}$$

where $T_i(x), U_i(x)$ are Chebyshev polynomials of the first and second kind respectively. We assume that $g(x)$ be a real-valued function, so the Fourier coefficients a_i, b_i are determined as follows:

$$a_i = \frac{2}{\pi}\int_{-1}^{1} g(x)T_i(x)\frac{dx}{\sqrt{1-x^2}}, \qquad i = 0,1,2,3,\ldots, \qquad x \in [-1,1], \tag{2.42}$$

$$b_i = \frac{2}{\pi}\int_{-1}^{1} g(x)U_i(x)\sqrt{1-x^2}dx, \qquad i = 0,1,2,3,\ldots, \qquad x \in [-1,1]. \tag{2.43}$$

Many boundary value problems in mathematical physics, engineering and mechanics can be reduced to the LSIEs. For examples see Lifanov (1996), Chan et al. (2003) Chakrabarti & Manam (2003) and Kabir et al. (1998).