A numerical solution for Duffing-Van Der Pol oscillators using a backward difference formulation

Cite as: AIP Conference Proceedings **2016**, 020120 (2018); https://doi.org/10.1063/1.5055522
Published Online: 27 September 2018

Ahmad Fadly Nurullah Rasedee, Mohammad Hasan Abdul Sathar, Hazizah Mohd Ijam, Khairil Ikandar Othman, Norizarina Ishak, and Siti Raihana Hamzah

A numerical solution for Duffing-Van Der Pol oscillators using a backward difference formulation

Ahmad Fadly Nurullah Rasedee[1, 2, a)], Mohammad Hasan Abdul Sathar[2], Hazizah Mohd Ijam[2], Khairil Ikandar Othman[3], Norizarina Ishak[4], Siti Raihana Hamzah[4].

[1]*Faculty of Economics and Muamalat, Universiti Sains Islam Malaysia, 71800 Nilai, Malaysia.*
[2]*Institute for Mathematical Research, Universiti Putra Malaysia, 43400 Serdang, Malaysia.*
[3]*Faculty of Computer and Mathematical Sciencecs, Universiti Teknologi MARA, 40450 Shah Alam, Malaysia.*
[4]*Faculty of Science and Technology, Universiti Sains Islam Malaysia, 71800 Nilai, Malaysia.*

[a)]Corresponding author: fadlynurullah@usim.edu.my

Abstract. The study of chaotic motion in periodic self-excited oscillators are an area of interest in science and engineering. In the current research, a numerical solution in backward difference form is proposed for solving these chaotic motions in periodic- self excited oscillators. Study conducted in this article focuses on chaotic motions in the form of Duffing-Van Der Pol Oscillators. A backward difference formulation in predictor-corrector (PeCe) mode is introduced for solving these Duffing-Van Der Pol directly. Numerical simulations provided will show the accuracy of the PeCe backward difference formulation.

Keywords: Backward difference, ODEs, Duffing-Van Der Pol oscillators.

INTRODUCTION

Previous studies (see[1]) shows that the study of chaotic phenomena in the area of nonlinear periodic self-excited oscillators has been an area of attraction by researchers especially, the Duffing-Van Der Pol oscillator. The general Duffing-Van Der Pol oscillator is governed by the following second order ordinary equation (ODE)

$$y'' - \mu\left(1-t^2\right)y' + \omega_0 t + \beta t^3 = f\left(t\right) \tag{1}$$

where $\mu > 0$, ω_0 and β are constants and $f(t)$ is a function of time. The Duffing-Van Der Pol oscillator is commonly used to describe electrical circuits with many applications in science and engineering.

A popular approach for obtaining numerical solution of second order ODEs were conducted by authors in [2], [3] and [4]. Based on fundamental ideas adopted from [4], research for direct solution of second order ODEs was continued by [5], [6], [7], [8], [9] and many more. Suleiman [4], designed a divided difference code for solving stiff and non-stiff higher order ODEs directly. This divided difference code was referred as the Direct Integration (DI) method. The downside of the DI method is the demanding calculations required by the divided differences formulation which requires re-computing the integration coefficients at every step change. Implementing a backward difference formulation produces a more elegant error formulae compared to the divided difference which allows for an added advantage when formulating and coding the method. In the next section, the formulation of the backward difference coefficient is provided.

Proceedings of the 3rd International Conference on Applied Science and Technology (ICAST'18)
AIP Conf. Proc. 2016, 020120-1–020120-6; https://doi.org/10.1063/1.5055522
Published by AIP Publishing. 978-0-7354-1734-2/$30.00

Derivation of the Explicit and Implicit Backward Differences Method for Second Order ODEs

For the specific purpose of this study, we consider an initial value problem (IVP) in the form of a second order ODE,

$$y'' = f\left(t, \tilde{Y}\right), \tag{2}$$

with $\tilde{Y}(\alpha) = \tilde{\eta}$ in the interval $\alpha \le t \le \beta$, where

$$\tilde{Y}(t) = (y, y') \text{ and } \tilde{\eta}(t) = (\eta, \eta')$$

Integrating the second order ODE (2) once gives

$$y'(t_{n+1}) = y'(t_n) + \int_{t_n}^{t_{n+1}} f(y, y')\,dt.$$

Newton-Gregory backward difference interpolation polynomial, $P_n(t)$ as

$$P_n(t) = \sum_{i=0}^{k-1} (-1)^i \binom{-s}{i} \nabla^i f_n, \qquad s = \frac{t - t_n}{h}.$$

which interpolates $f(y, y')$ at k back values.

Next, $f(y, y')$ is approximated by $P_n(t)$ and by changing the limit of integration gives

$$y'(t_{n+1}) = y'(t_n) + h\sum_{i=0}^{k-1} \gamma_{1,i} \nabla^i f_n \; ds, \qquad \gamma_{1,i} = (-1)^i \int_0^1 \binom{-s}{i} ds. \tag{3}$$

Let $G_1(t)$ be the generating function for the coefficients $\gamma_{1,i}$ and defined as follows:

$$G_1(t) = \sum_{i=0}^{\infty} \gamma_{1,i} t^i$$

By substituting $\gamma_{1,i}$ from (3) in $G_1(t)$ and through mathematical induction, the generating function can be denoted as

$$G_1(t) = -\left[\frac{(1-t)^{-1}}{log(1-t)} - \frac{1}{log(1-t)}\right].$$

This this gives the backward difference coefficients formulation, $\gamma_{1,k}$

$$\gamma_{1,0} = 1, \quad \gamma_{1,k} = 1 - \sum_{i=0}^{k-1} \left(\frac{\gamma_{1,i}}{k-i+1}\right), \quad k = 1, 2, \ldots.$$

Then, integrating (2) twice and repeating the same mathematical process as above, we obtain

$$y(t_{n+1}) = y(t_n) + hy'(t_n) + h^2 \sum_{i=0}^{k-1} \gamma_{2,i} \nabla^i f_n \; ds.$$

Thus, it can be perceived that the second order generating functions, $G_2(t)$ and $G_2^*(t)$ can be written similar to $G_1(t)$ as

$$G_2(t) = \left[\frac{1}{log(1-t)} - \frac{G_1(t)}{log(1-t)}\right], \quad G_2^*(t) = \left[\frac{(1-t)}{log(1-t)} - \frac{G_1^*(t)}{log(1-t)}\right].$$

Replicating calculations obtained in [6] the relationship between the second order explicit and implicit coefficients are given by

$$\sum_{i=0}^{k} \gamma_{d,i}^{*} = \gamma_{d,k}, \quad d = 1,2.$$

The following are examples a few explicit and implicit integration coefficients and their coding algorithm.

TABLE 1. Explicit and Implicit coefficients for k from 0 to 5.

k	**0**	**1**	**2**	**3**	**4**	**5**
$\gamma_{1,k}$	1	$\frac{1}{2}$	$\frac{5}{12}$	$\frac{3}{8}$	$\frac{251}{720}$	$\frac{95}{288}$
$\gamma_{2,k}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{19}{180}$	$\frac{3}{32}$	$\frac{683}{10080}$
$\gamma_{1,k}^{*}$	1	$-\frac{1}{2}$	$-\frac{1}{12}$	$-\frac{1}{24}$	$-\frac{19}{720}$	$-\frac{3}{160}$
$\gamma_{2,k}^{*}$	$\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{24}$	$-\frac{7}{360}$	$-\frac{17}{1440}$	$-\frac{41}{5040}$

Algorithm 1: Integration coefficients.

```
for(I=0;I<=12;I++)
{
                G[0][I]=pow(1,I);
}
for(M=1;M<=D[1];M++)
{
        for (I= 0; I<= 15-M;I++)
        {
                if (M==1)
                {
                        TEMP = 1.0;
                }
                else
                {
                        TEMP=G[M-1][I+1];
                }
                for (T=0;T<=I-1;T++)
                {
                        TEMP=TEMP-(G[M][T]/(I+1.0-T));
                }
                G[M][I]=TEMP;
                if(I==0) G1[M][0]=G[M][0];
                else G1[M][I]=G[M][I]- G[M][I-1];

        }
}
```

NUMERICAL RESULTS

The vast application of Duffing-Van Der Pol equations in science and engineering, especially in problems relating to electrical circuits are among the reason it has been studied by authors such as [1], [10], [11], [12] and [13]. In the current research, the proposed backward difference method (BDF) is used to numerically approximate a few popular Duffing-Van Der Pol type equations. The accuracy of the BDF method is tested against known methods that have been shown to be efficient.

H: step size,
LM: Lindsteds method,
ADM: Adomians Decomposition method,
HPM: Homotopy Perturbation Method,
DTM: Differential Transform method,
RK4: Fourth Order Runge Kutta method,
DI: Direct Integration method,
1PBDF: 1 Point Backward Difference method.

Duffing-Van Der Pol Problem 1: [11]

$$y''(t) = -\left(\frac{4}{3} - 3y^2(t)\right)y'(t) - \frac{1}{3}y(t) - y(t)^3 = 0,,$$

with intial conditions

$$y(0) = -0.28868, \quad y'(0) = 0.12. \; .$$

Duffing-Van Der Pol Problem 2: [12]

$$y''(t) = 0.1\left(1 - y^2(t)\right)y'(t) - y(t) - 0.01y(t)^3 = 0,,$$

with intial conditions

$$y(0) = 2, \quad y'(0) = 0. \; .$$

Duffing-Van Der Pol Problem 3: [13]

$$y''(t) = cos(0.7t) + \left(1 - y^2(t)\right)y'(t) + y(t) - y(t)^3 = 0,$$

with intial conditions

$$y(0) = 0.1, \quad y'(0) = -0.2.$$

TABLE 2. Numerical Results for Duffing-Van Der Pol Problem 1.

t	RK4	HPM	DTM	1PBDF
0.1	-0.28748349253	-0.28748347499	-0.28748522585	**-0.28748347499**
0.2	-0.28629387437	-0.28629385687	-0.28630661206	**-0.28629385687**
0.3	-0.28511109904	-0.28511108168	-0.28514428684	**-0.285111081680**
0.4	-0.28393510355	-0.28393508631	-0.28399838456	**-0.28393508631**
0.5	-0.28276582565	-0.28044716121	-0.28286904593	**-0.28276580855**
0.6	-0.28160320408	-0.27929767140	-0.28175641828	**-0.28160318703**
0.7	-0.28044717833	-0.27814565867	-0.28066065588	**-0.28044716121**
0.8	-0.27929768854	-0.27701806494	-0.27951892013	**-0.27929767140**
0.9	-0.27815467572	-0.27815465866	-0.27815465867	**-0.27815465867**
1.0	-0.27701808173	-0.27701806494	-0.27747621187	**-0.27701806494**

TABLE 3. Numerical Results for Duffing-Van Der Pol Problem 2.

t	LM	ADM	1PBDF
0.0	2.00000	1.99750	**2.00000**
0.1	1.98971	1.98724	**1. 98971**
0.2	1.95936	1.95697	**1.95936**
0.3	1.90980	1.90758	**1.90980**
0.4	1.84202	1.84008	**1.84202**
0.5	1.75702	1.75552	**1.75702**
0.6	1.65586	1.65493	**1.65586**
0.7	1.53958	1.53937	**1.53958**
0.8	1.40922	1.40982	**1.40923**
0.9	1.26581	1.26726	**1.26586**
1.0	1.11033	1.11267	**1.11054**

TABLE 4. Numerical Results for Duffing-Van Der Pol Problem 3.

t	DI	1PBDF		
	$H = 0.1$	$H = 0.1$	$H = 0.001$	$H = 0.00001$
0.5	0.13038	**0.13041**	**0.13047**	**0.13047**
1.0	0.52399	**0.52395**	**0.52428**	**0.52428**
1.5	1.33649	**1.33602**	**0.52428**	**0.52428**
2.0	1.74809	**1.74955**	**1.74814**	**1.74814**
2.5	1.42639	**1.42773**	**1.42613**	**1.42613**
3.0	0.88625	**0.88643**	**0.88594**	**0.88594**
3.5	0.16497	**0.16388**	**0.16456**	**0.16456**
4.0	-1.12614	**-1.12754**	**-1.12676**	**-1.12676**
4.5	-2.08061	**-2.08416**	**-2.08006**	**-2.08006**

DISCUSSION AND CONCLUSION

The Duffng-Van Der Pol oscillator is a second order ODE with high frequency oscillations. Problem1-3 consist of known and application problems in the form of Duffng-Van Der Pol equations. TABLE 2 shows the numerical results for Problem 1 where the 1PBDF method is compared against Lindsted's method (LM) and a modified Adomians Decomposition method (ADM). The results show that the 1PBDF method has better accuracy than the ADM and as is competitive as the LM. In TABLE 3, the accuracy of the 1PBDF method is tested against the fourth order Runge-Kutta method (RK4), Homotopy Perturbation Method (HPM) and differential Transform method (DTM) and shown to as accurate. Finally, TABLE 4 Illustrates the numerical results of the 1PBDF method with another multistep method, the Direct Integration (DI) method. The 1PBDF method was tested with Problem 3 using 3 different step sizes to show the consistency of the method. Unfortunately, we were only able to provide numerical results for the DI method using only 1 step size. This is because when using smaller step sizes, the DI was unable to compute the approximated solution. This set back is most likely due to the division component of the DI method. This issue had previously been addressed in [14] when attempting to solve a type Van Der Pol equation. By justification of the numerical results, the 1PBDF has shown to be a viable method for solving Duffng-Van Der Pol type oscillatory problems.

ACKNOWLEDGMENTS

This research conducted by this article has been supported by Universiti Putra Malaysia under Grant Putra (GP), project number GP-IPS/2017/9526600, Ministry of Higher Education (MoHE) under the Fundamental Research Grant Scheme (FRGS), project number USIM/FRGS/FEM/055002/51517 and Universiti Sains Islam Malaysia (USIM) under the Short-Term Grant Scheme, project number PPP/FST-0117/051000/11417.

REFERENCES

1. L. Zhou and F. Chen, Shock and Vibration 2014 (2014).
2. G. Hall and J. M. Watt, *Modern numerical methods for ordinary differential equations* (Clarendon Press, Oxford, 1976).
3. G. Hall and M. B. Suleiman, IMA: Journal of Numerical Analysis **1**, 427–438 (1981).
4. M. B. Suleiman, Applied Mathematics and Computation **33**, 197–219 (1989).
5. Z. B. Ibrahim, M. Suleiman, and K. I. Othman, World Academy of Science, Engineering and Technology 1, 120–122 (2009).
6. M. B. Suleiman, Z. B. Ibrahim and A. F. N. Rasedee, Mathematical Problems in Engineering **2011**, Article ID 810324, p. 18 pages (2011).
7. H. Mohd Ijam, M. Suleiman, A. F. N. Rasedee, N. Senu, A. Ahmadian, and S. Salahshour, Abstract and Applied Analysis, **Vol. 2014**, Article ID 867095, p. 13 pages (2014).
8. A. F. N. Rasedee, N. Ishak, S. R. Hamzah, H. M. Ijam, M. Suleiman, Z. B. Ibrahim, M. H. A. Sathar, N. A. Ramli, and N. S. Kamaruddin, “Variable order variable stepsize algorithm for solving nonlinear Duffng oscillator,” in Journal of Physics: Conference Series, Vol. 890 (IOP Publishing, 2017) p. 012045.
9. A. F. N. Rasedee, MATEMATIKA 33, 165–176 (2017).
10. S. Mukherjee, B. Roy, and S. Dutta, Physica Scripta 83, p. 015006 (2010).
11. N. A. Khan, M. Jamil, S. A. Ali, and N. A. Khan, International Journal of Differential Equations 2011 (2011).
12. G. A. Cordshooli and A. Vahidi, Journal of Physics 3, 121–129 (2011).
13. C. Zhang and Y. Zeng, SRM 1, p. 3 (2014).
14. A. F. N. Rasedee, M. b. Suleiman, and Z. B. Ibrahim, Mathematical Problems in Engineering 2014 (2014).