

CHAPTER 3

EXTENDED CHEBYSHEV POLYNOMIALS OF THE FIRST AND SECOND KINDS OF ORDER k

3.1 Introduction

In this chapter, we propose a new class of orthogonal polynomials named Extended Chebyshev polynomials (ECPs) of the first and second kinds of order k , denoted as $Z_{(i,n)}^k(x)$, $i = \{1, 2\}$, k is positive odd integers; $n = 0, 1, 2, \dots$, which is an extension of the Chebyshev polynomials of the first and second kinds respectively. The zeros of ECPs of the first and second kinds of order k are discussed and described. The main characteristics behavior of Extended Chebyshev sequences $\{Z_{(i,n)}^{2m+1}(x)\}_{m=0}^k$; $i = \{1, 2\}$; $n = 0, 1, 2, 3, \dots$ are described when k is sufficient large odd integers. Moreover we focus on the construction of recurrence relations and the relationships between each other.

We state some of the standard formulae of the two kinds. Additionally, we focus on orthogonality properties of ECPs of the first and second kinds of order k , $(Z_{(i,n)}^k(x), i = \{1, 2\})$, it was found that ECPs of the first and second kinds of order k $Z_{(i,n)}^k(x)$, $i = \{1, 2\}$ are orthogonal with respect to weight functions $w_{(1,k)}(x) = \frac{x^{k-1}}{\sqrt{1-x^{2k}}}$, $w_{(2,k)}(x) = x^{k-1}\sqrt{1-x^{2k}}$ respectively on the interval $[-1, 1]$, where k is positive odd integers.

Spectral properties of ECPs of the first and second kinds of order k are proved. They will be used in greater detail in this chapter. Moving into the applications of ECPs of the first and second kinds of order k , we will demonstrate how to use these orthogonal polynomials to aid in the solution of SIEs, namely:

- Special class of logarithmic singular integral equations of order k (LogSIEk).
- First kind special class of singular integral equations of order k (SSIEk).

Eigenvalues and corresponding eigenfunctions are found for the LogSIEk of the first kind, for inhomogeneous LogSIEk the approximate solution are found. For linear SSIEk of the first kind, the unknown functions are characterized in terms of truncated series of ECPs of the first and second kinds of order k into two cases:

- Case (I): The solution is unbounded at both end points $x = \pm 1$,
- Case (II): The solution is bounded at both end points $x = \pm 1$.

3.2 Definitions of Extended Chebyshev Polynomials of the First and Second kinds and their Properties

Let us first consider two definitions of ECPs of the first and second kinds of order k , $Z_{(i,n)}^k(x)$, $i = \{1, 2\}$ and some properties of them, where k is positive odd integers. The condition of k is positive odd integers are necessary, on the contrary. If we assume that k is positive even integers, then many properties will be lost. Moreover the integral of Extended Chebyshev polynomials with weight functions reduces to zero.

Definition 9 *Extended Chebyshev polynomials of the first kinds of order k : Extended Chebyshev polynomials of the first kinds of order k , $(Z_{(1,n)}^k(x))$, where k is positive odd integer, is a polynomial in x of degree (kn) , defined by the relation (for fixed k):*

$$Z_{(1,n)}^k(x) = \cos(n\theta), \quad x^k = \cos(\theta), \quad k = 1, 3, 5, \dots, N; \quad n = 0, 1, 2, \dots \quad (3.1)$$

Definition 10 *Extended Chebyshev polynomials of the second kind of order k : Extended Chebyshev polynomials of the second kind of order k , $(Z_{(2,n)}^k(x))$ where k is positive odd integer, is a polynomials in x of degree (kn) , defined by the relation (for fixed k):*

$$Z_{(2,n)}^k(x) = \frac{\sin((n+1)\theta)}{\sin \theta}, \quad x^k = \cos(\theta) \quad (3.2)$$

where k is positive odd integers; $n = 0, 1, 2, \dots$

The range of the variable x is $[-1, 1]$ and the range of the corresponding variable θ is $[0, \pi]$. These ranges are traversed in opposite directions i.e. when $x = -1$, corresponds to $\theta = \pi$ and $x = 1$ corresponds to $\theta = 0$. The above definitions are well defined for $x \in [-1, 1]$ with $|\cos(\theta)| \leq 1$, implies $|x|^k \leq 1$.

The relationship between Extended Chebyshev (ECPs) and Chebyshev polynomials are:

$$Z^k_{(i,n)}(x) = \begin{cases} T_n(x^k), & i = 1, \\ U_n(x^k), & i = 2, \end{cases} \tag{3.3}$$

where $T_n(x), U_n(x)$ are Chebyshev polynomials of the first and second kinds respectively, k is positive odd integers, $n = 0, 1, 2, 3 \dots$, the proof directly comes by letting $u = x^k$, and a special case for $k = 1$, we have:

$$Z^1_{(i,n)}(x) = \begin{cases} T_n(x), & i = 1, \\ U_n(x), & i = 2, \end{cases} \tag{3.4}$$

where $T_n(x)$ and $U_n(x)$ are Chebyshev polynomials of the first and second kinds respectively, $n = 0, 1, 2, 3 \dots$.

Table 3.1 refers to the first few ECPs of first and second kinds of order k , $Z^k_{(i,n)}(x), i = \{1, 2\}, n = 0, 1, 2, \dots$ respectively.

Table 3.1: Extended Chebyshev polynomials of the first and second kinds

n	$Z^k_{(1,n)}(x) = T_n(x^k)$	$Z^k_{(2,n)}(x) = U_n(x^k)$
0	1	1
1	x^k	$2x^k$
2	$2x^{2k} - 1$	$4x^{2k} - 1$
3	$4x^{3k} - 3x^k$	$8x^{3k} - 4x^k$
4	$8x^{4k} - 8x^{2k} + 1$	$16x^{4k} - 12x^{2k} + 1$
5	$x^k(16x^{4k} - 20x^{2k} + 5)$	$32x^{5k} - 32x^{3k} + 6x^k$
6	$32x^{6k} - 48x^{4k} + 18x^{2k} - 1$	$64x^{6k} - 80x^{4k} + 24x^{2k} - 1$

ECPs are transformation of Chebyshev polynomials, smoothly and simply. Indeed, polynomial functions can be used to model science and describe real-world phenomena. In addition, another reason for their advantages is in analysis and other science branches. In particular, among many polynomial functions properties appeared in approximation theory; every continuous function defined on a compact interval can be approximated by polynomials. In this sense, this family of polynomials are rich and useful.

3.2.1 Plotting of Extended Chebyshev polynomials of the First and Second Kinds

Figures 3.1 - 3.5, we show the respective graphs of the Extended Chebyshev polynomial of the first kind $Z_{(1,n)}^k(x)$, $n = 1, 2, 3, 4$, $k = 3$, starting with a value of -1 and finishing with a value of 1 , since when $x = 1 \implies \cos \theta = 1$, that is mean $\theta = 0 \implies \cos n0 = 1$ yield $Z_{(1,n)}^k(1) = 1$ for any $n = 0, 1, 2, 3, \dots$.

On the other hand, when $x = -1 \implies \cos \theta = -1 \implies \theta = \pi$, and hence the ECPs of the first kind has two values at $x = -1$ depends on n :

$$Z_{(1,n)}^k(-1) = \cos(n\pi) = \begin{cases} -1, & n = \text{odd}, \\ 1, & n = \text{even}, \end{cases} \quad (3.5)$$

where k is positive odd integers.

The Extended Chebyshev polynomials of the first kind are continuous on $[-1, 1]$ and differentiable on $(-1, 1)$ and therefore integrable. ECPs of the first kind have n zeros in $[-1, 1]$; indeed, these zeros are related with value of k (see Eq. 3.8). In particular, it will be noted that the shape of $Z_{(1,n)}^k(x)$ on $[-1, 1]$ is oscillating between extreme points with equal magnitudes, see Figures 3.1 - 3.6. According to the definition ECPs of the first kind (Definition 9), the zeros for x in $[-1, 1]$ of $Z_{(1,n)}^k(x)$ must correspond to the zeros for θ in $[0, \pi]$ of $\cos n\theta$, i.e:

$$\cos(n\theta) = 0$$

This means

$$n\theta = (m - \frac{1}{2})\pi, \quad m = 1, 2, 3, \dots, n \quad (3.6)$$

We have the zeros of $Z_{(1,n)}^k(x)$:

$$T_n(x^k) = 0; \quad n = 1, 2, 3, \dots, m, \quad (3.7)$$

$$x = x_{(1,m)} = \sqrt[k]{\cos \frac{(m - \frac{1}{2})\pi}{n}}; \quad m = 1, 2, 3, \dots, n. \quad (3.8)$$

where $x_{(1,m)}$ denoted that the zeros of ECPs of the first kind of order k .

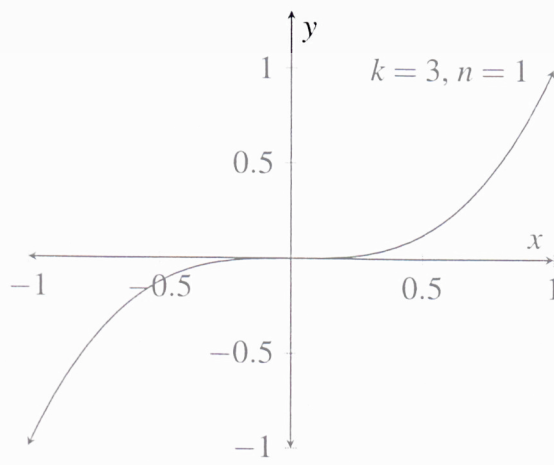


Figure 3.1: Plot of ECP of the first kind of order 3, $Z_{(1,1)}^3(x)$

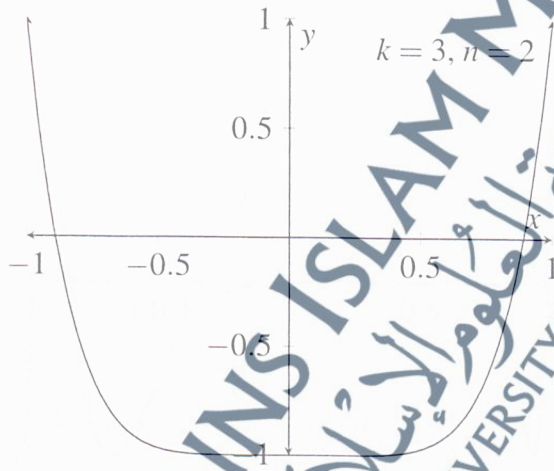


Figure 3.2: Plot of ECP of the first kind of order 3, $Z_{(1,2)}^3(x)$

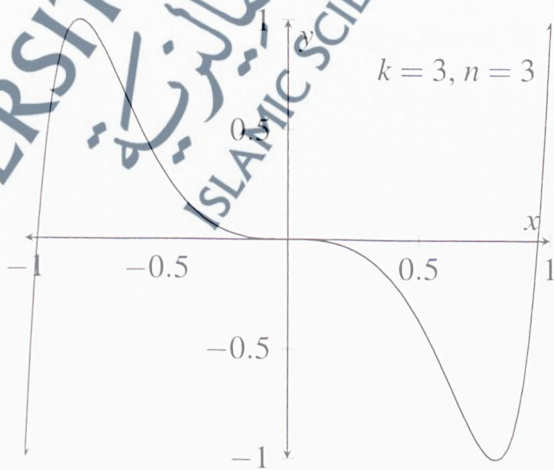


Figure 3.3: Plot of ECP of the first kind of order 3, $Z_{(1,3)}^3(x)$

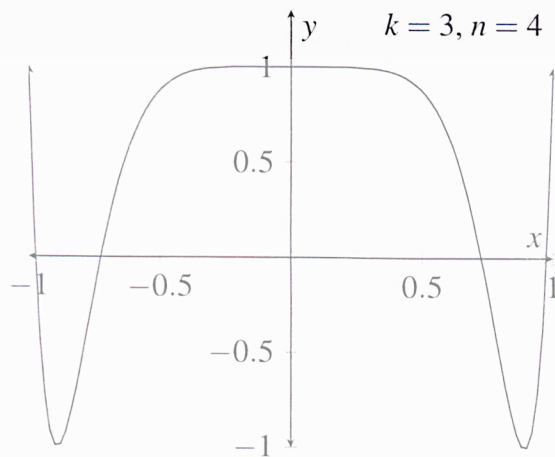


Figure 3.4: Plot of ECP of the first kind of order 3, $Z_{(1,4)}^3(x)$

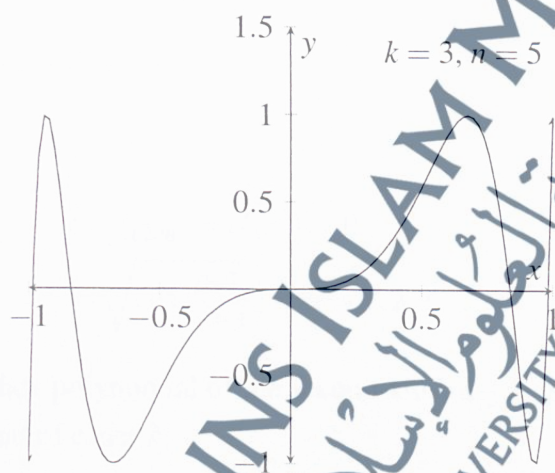


Figure 3.5: Plot of ECP of the first kind of order 3, $Z_{(1,5)}^3(x)$

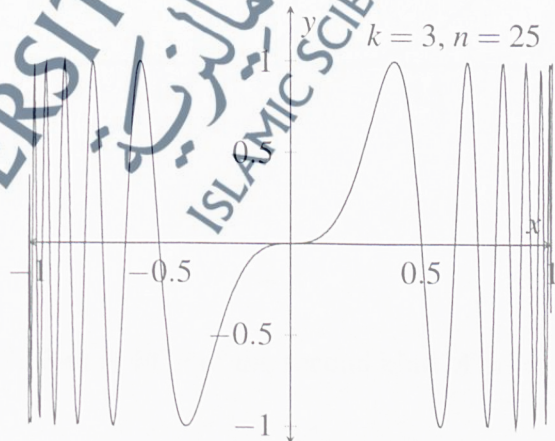


Figure 3.6: Plot of ECP of the first kind of order 3, $Z_{(1,25)}^3(x)$

The ECPs of the second kind ($Z_{(2,n)}^3(x)$) are plotted when $n = 1, 2, 3, 4, 5$ in Figures 3.7 - 3.14. It is clear that ECPs of the second kind like the first kind, it is an even or odd function, involving only even or odd powers of x , according as $(n \times k)$ is even or odd.

The ECPs of the second ($Z_{(2,n)}^k(x)$) kind has precisely n zeros in the interval $[-1, 1]$ (Figures 3.7 - 3.14). The definition of ECPs of the second kind (definition 10) shows that the zeros of ($Z_{(2,n)}^k(x)$) for x in $[-1, 1]$ are determined in a similar way from the zeros of $\sin(n+1)\theta$ as:

$$\begin{aligned} \sin(n+1)\theta &= 0, \\ \theta &= \frac{m\pi}{n+1}, \quad m = 1, 2, 3, \dots, n. \end{aligned}$$

We get the zeros of $Z_{(2,n)}^k(x)$:

$$\begin{aligned} x &= x_{(2,m)} = U(x^k) = 0 \\ &= \sqrt[k]{\cos \frac{m\pi}{n+1}}, \quad m = 1, 2, 3, \dots, n \end{aligned} \tag{3.9}$$

where $U(x)$ is Chebyshev polynomial of the second kind, $x_{(2,m)}$ is denoted to the zeros of ECPs of the second kind of order k .

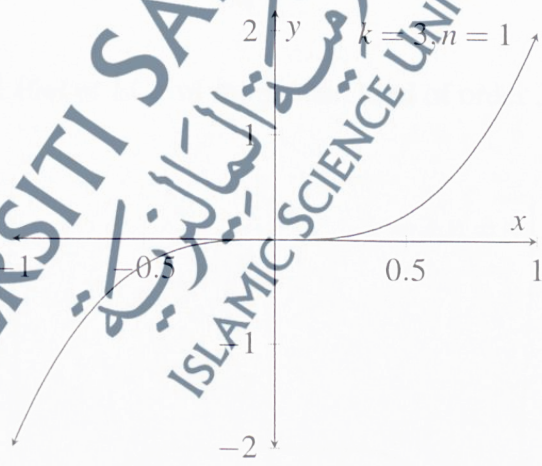


Figure 3.7: Plot of ECP of the second kind of order 3, $Z_{(2,1)}^3(x)$

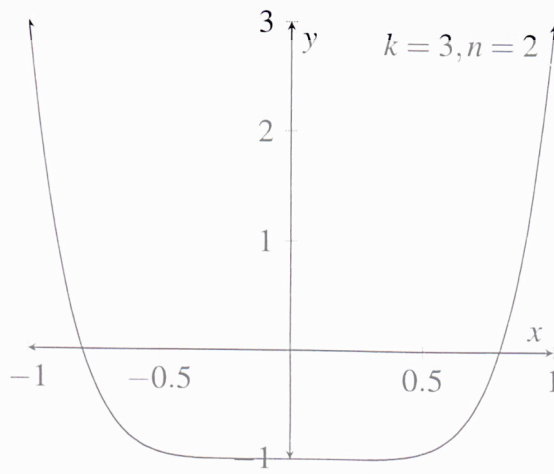


Figure 3.8: Plot of ECP of the second kind of order 3, $Z_{(2,2)}^3(x)$

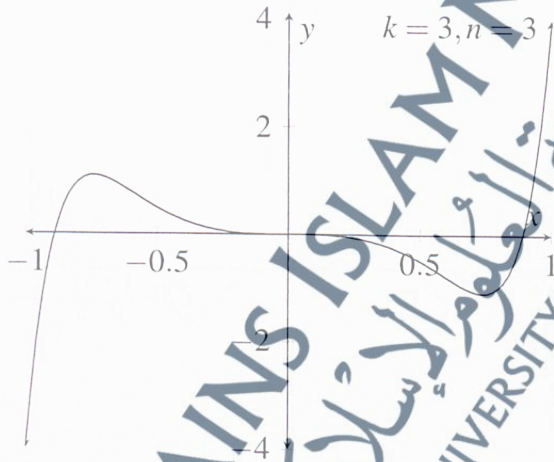


Figure 3.9: Plot of ECP of the second kind of order 3, $Z_{(3,3)}^3(x)$

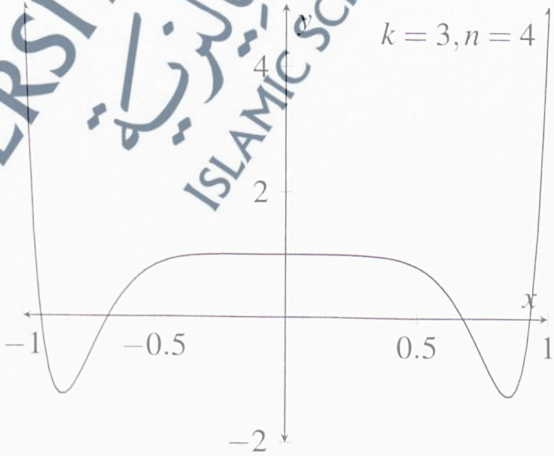


Figure 3.10: Plot ECP of the second kind of order 3, $Z_{(2,4)}^3(x)$

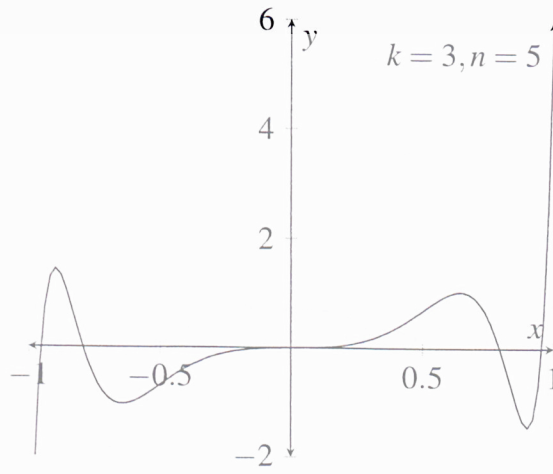


Figure 3.11: Plot of ECP of the second kind of order 3, $Z_{(2,5)}^3(x)$

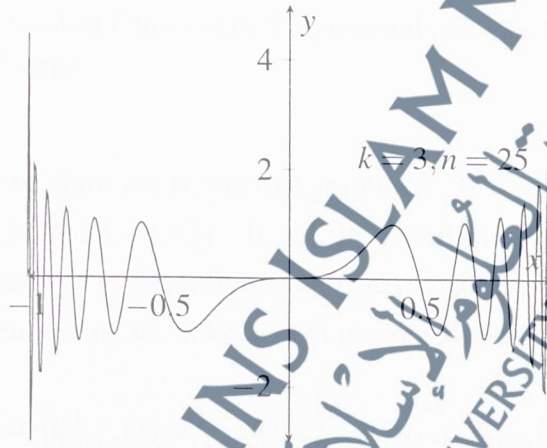


Figure 3.12: Plot of ECP of the second kind of order 3, $Z_{(2,25)}^3(x)$

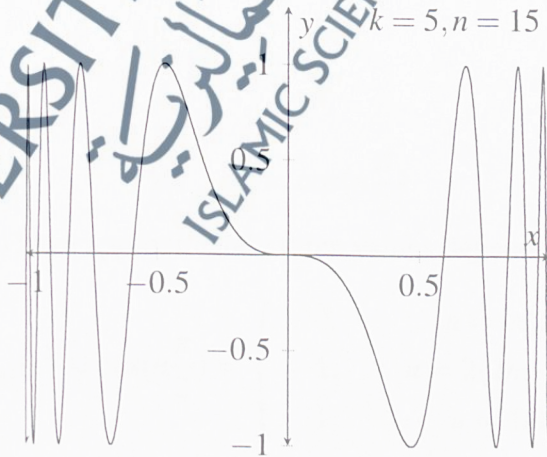


Figure 3.13: Plot of ECP of the first kind of order 5, $Z_{(1,15)}^5(x)$

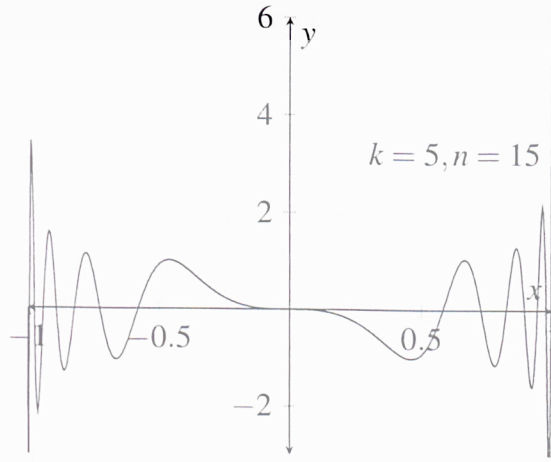


Figure 3.14: Plot of ECP of the second kind of order 5, $Z_{(2,15)}^5(x)$

3.2.2 Behavior of Extended Chebyshev Polynomials of The First and Second Kinds when k is Too Large

In Figures 3.15 - 3.17, we show the respective graphs of $Z_{(1,n)}^k(x)$, $n = 2, 3, 4$ and different orders of k , $k = \{1, 3, 5, 15, 25, 49, 63\}$. It will be noted that the shapes of ECP of the first kind on $[-1, 1]$ when k is sufficiently large positive odd integer is classified into three distinct categories depending on index n . Let us consider a finite sequence of ECPs of the first kind:

$$A = \{Z_{(1,n)}^1(x), Z_{(1,n)}^3(x), Z_{(1,n)}^5(x), \dots, Z_{(1,n)}^k(x)\} \quad (3.10)$$

where $n = 0, 1, 2, 3, \dots$ and assume $x \in (-1, 1)$, the term x^k becomes very small when k is too large, we assume that approaches to zero; this means $\cos \theta$ also approaches to zero $\implies \theta \rightarrow \frac{\pi}{2}$ and hence, the ECPs of the first kind with order sufficient large k can be approximate:

$$Z_{(1,n)}^k(x) \approx \cos(n\frac{\pi}{2}). \quad (3.11)$$

The ECPs in Eq. (3.11) have approximately three values dependent on index n :

$$Z_{(1,n)}^k(x) \approx \cos(n\frac{\pi}{2}) = \begin{cases} 0, & n = \text{odd}, \\ -1, & n = 2(2m+1), \\ 1, & n = 4m. \end{cases} \quad (3.12)$$

where k is a sufficient large positive odd integer, $m = 0, 1, 2, 3, \dots$

Figure 3.15 shows that the behavior Extended Chebyshev sequences when $n = 2$ with

different order of k , $\{k = 1, 3, 5, \dots, 11\}$. The behavior of Extended Chebyshev sequences converges to -1 which coincides to our expectations in Eq . (3.12). Figure 3.16 shows that the graph of Extended Chebyshev sequences when $n = 3$ (odd positive integers) with different order values of k , $\{k = 1, 3, 5, \dots, 11\}$, the behavior of ECPs converge to 0. Figure 3.17 shows that the graph of Extended Chebyshev sequences when $n = 2$ (has form $2(m + 1)$) with different order values of k , $\{k = 1, 3, 5, \dots, 11\}$, the behavior of Extended Chebyshev sequences converge to 1 which coincide to Eq . (3.12).

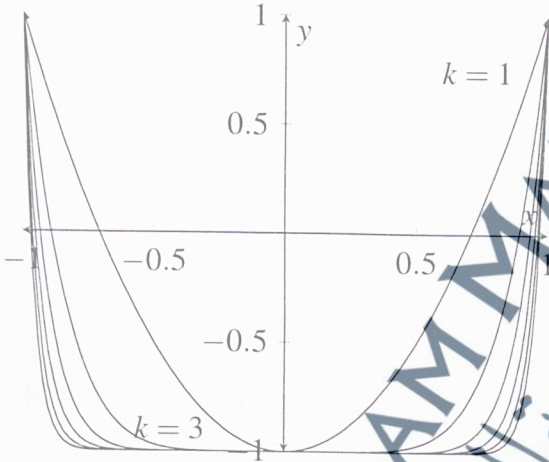


Figure 3.15: Plot of EC sequences of the first kind for different orders k , $Z_{(1,2)}^k(x)$

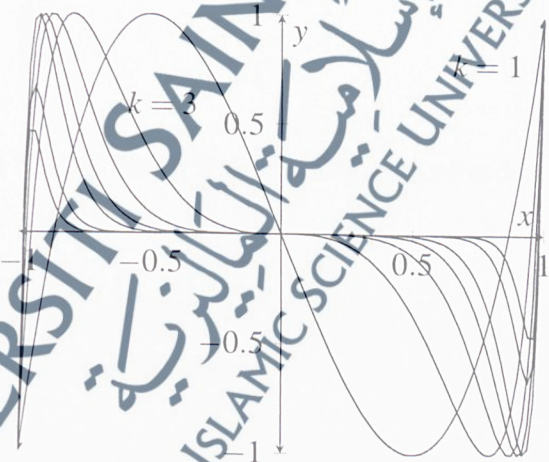


Figure 3.16: Plot of EC sequences of the first kind for different orders k , $Z_{(1,3)}^k(x)$

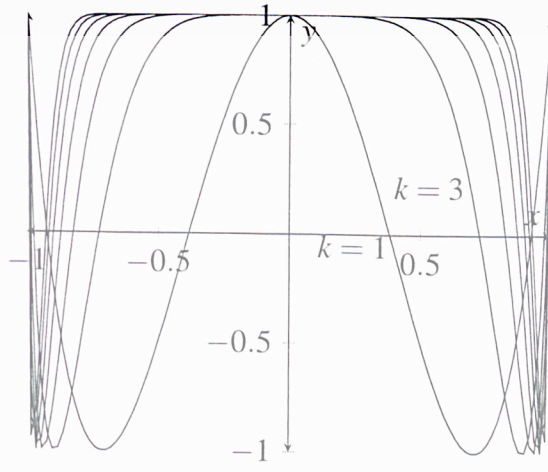


Figure 3.17: Plot of EC sequences of the first kind for different orders k , $Z_{(1,4)}^k(x)$

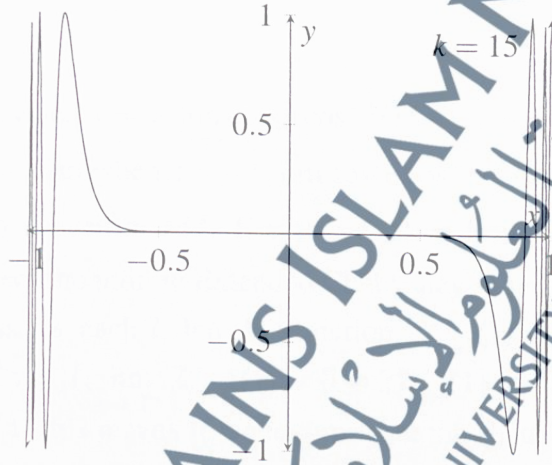


Figure 3.18: Plot of the ECP of the first kind of order 15, $Z_{(1,15)}^{15}(x)$

Moreover, if we assume that $k = 2l + 1, l = 0, 1, 2, 3, \dots$ and define an infinite sequence of Extended Chebyshev polynomial of the first kind such that:

$$A = \{Z_{(1,n)}^1(x), Z_{(1,n)}^3(x), Z_{(1,n)}^5(x), \dots, Z_{(1,n)}^{2l+1}(x), \dots\} \quad (3.13)$$

$$n = 0, 1, 2, 3, \dots$$

The behavior of an infinite Extended Chebyshev sequence A when $l \rightarrow \infty$ is classified into three categories:

- When $x = -1$

$$Z_{(1,n)}^{\infty}(-1) = \lim_{l \rightarrow \infty} Z_{(1,n)}^{2l+1}(-1) = \begin{cases} -1 & , n = \text{odd}, \\ 1 & , n = \text{even}. \end{cases} \quad (3.14)$$

- When $x = 1$

$$Z_{(1,n)}^{\infty}(1) = \lim_{l \rightarrow \infty} Z_{(1,n)}^{2l+1}(1) = 1 \quad (3.15)$$

- When $x \in (-1, 1)$

$$Z_{(1,n)}^{\infty}(x) = \cos(n\frac{\pi}{2}) = \begin{cases} 0, & n = \text{odd}, \\ -1, & n = 2(2m+1), \\ 1, & n = 4m. \end{cases} \quad (3.16)$$

$$m = 0, 1, 2, 3, \dots$$

It should be noted that when $x = 1$, $\lim_{l \rightarrow \infty} \cos(n \cos^{-1}(1^{2l+1})) = \cos(n0) = 1$ since $\cos(x)$ is a continuous function, also when $x = -1$, $\lim_{l \rightarrow \infty} \cos(n \cos^{-1}((-1)^{2l+1})) = \cos(n\pi) = \pm 1$ depending on the index n even or odd. If $x \in (-1, 1) \Rightarrow \lim_{l \rightarrow \infty} x^{2l+1} = 0, \forall x \in (-1, 1) \Rightarrow Z_{(1,n)}^{\infty}(x) = \cos(n\frac{\pi}{2})$. Here the infinite Extended Chebyshev sequence A contains sequence of continuous functions for each l , but the function $Z_{(1,n)}^{\infty}(x)$ is discontinuous function (precisely at end point $x = 1$, $\lim_{x \rightarrow 1^+} Z_{(1,n)}^{\infty}(x) = 0 \neq Z_{(1,n)}^{\infty}(1) = 1, n$ is odd, also the limit does not exist at $x = -1$, this means if we assume that the sequence of ECPs of the first kind formulates a converge Cauchy sequence, then this sequence has incomplete on the space of continuous function $(C[-1, 1])$.

Figures 3.19 - 3.21 show the respective graphs of $Z_{(2,n)}^k(x), n = 2, 3, 4$ with different orders of $k, \{k = 1, 3, 9, 25, 49, 63\}$. Indeed, the shapes of Extended Chebyshev sequences of the second kind on $[-1, 1]$ when k is sufficient large odd integers are classified into three distinct categories depends on the index n . Let us describe Extended Chebyshev sequences of the second kind $Z_{(2,n)}^k(x)$ and consider the finite sequence of ECP of the second kind:

$$B = \{Z_{(2,n)}^1(x), Z_{(2,n)}^3(x), Z_{(2,n)}^5(x), \dots, Z_{(2,n)}^k(x)\}. \quad (3.17)$$

where k is large positive odd integers, $n = 0, 1, 2, 3, \dots$ and assume $x \in (-1, 1)$, the term x^k becomes very small when k is too large, we assume that approach to zero $\Rightarrow \cos \theta$ also approaches to zero i.e $\cos \theta \rightarrow 0 \Rightarrow \theta \rightarrow \frac{\pi}{2}$, so the Extended Chebyshev sequences of the second kind when k is sufficient large odd integers can be approximate as:

$$Z_{(2,n)}^k(x) \approx \sin((n+1)\frac{\pi}{2}), \quad x \in (-1, 1). \quad (3.18)$$

The Extended Chebyshev sequences in Eq. (3.18) has approximately converge into one of the three values depending on index n

$$Z_{(2,n)}^k(x) \approx \sin((n+1)\frac{\pi}{2}) = \begin{cases} 0, & n = \text{odd}, \\ -1, & n = 2(2m+1), \\ 1, & n = 4m. \end{cases} \quad (3.19)$$

where k is sufficient large positive odd integer, $m = 0, 1, 2, 3, \dots$

Figures 3.19-3.21 show that the behavior of Extended Chebyshev sequences of the second kind when $n = 1, 2, 3$ respectively, with different orders of k , $\{k = 1, 3, 9, 15, 25, 49, 69\}$, the behavior of Extended Chebyshev sequences converge to one of the three values $\{0, -1, 1\}$ as in Eq. (3.19).

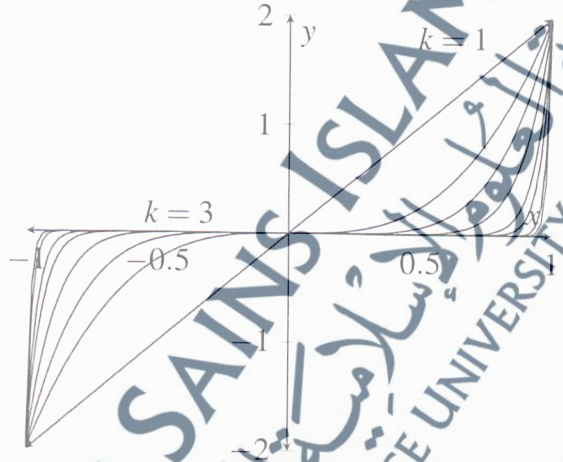


Figure 3.19: Plot of EC sequences of the second kind for different orders k , $Z_{(2,1)}^k(x)$

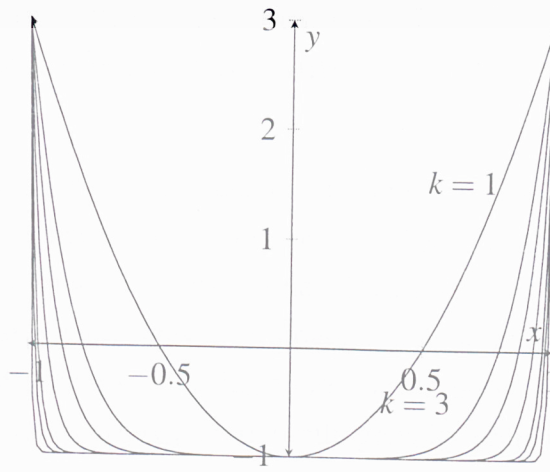


Figure 3.20: Plot of EC sequences of the second kind for different orders k , $Z_{(2,2)}^k(x)$

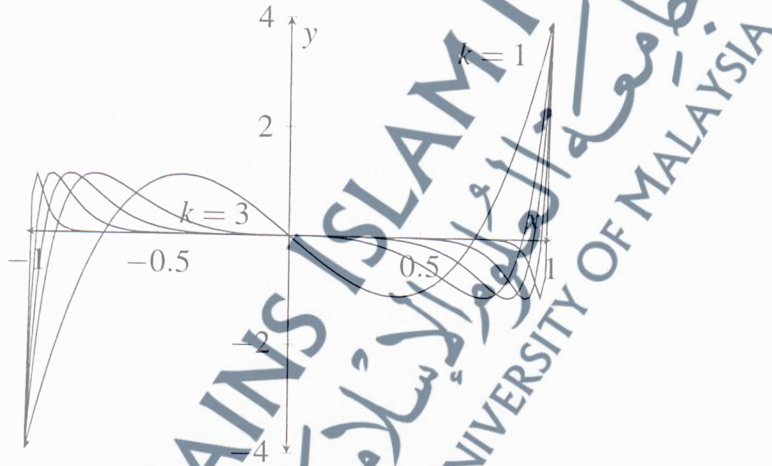


Figure 3.21: Plot of EC sequences of the second kind for different orders k , $Z_{(3,3)}^k(x)$

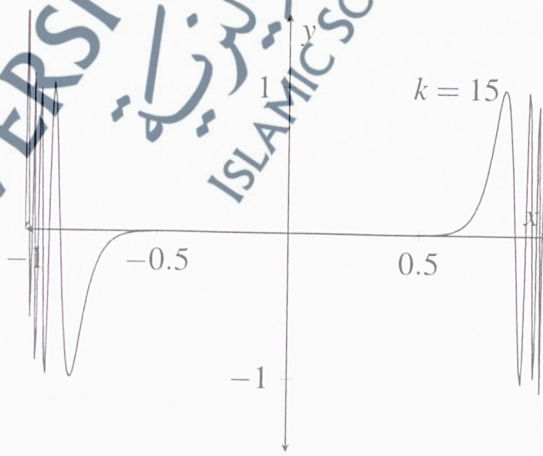


Figure 3.22: Plot of ECP of the second kind of order 15, $Z_{(2,35)}^{15}(x)$

3.3 Extended Chebyshev Polynomials of The First and Second Kinds of Order k and Weight Functions

Orthogonal polynomials play an important role in numerical analysis and functional space, mainly because functions belonging to very general classes can be expanded in the sense of orthogonal functions, such as, Fourier series, Fourier-Bessel series etc. An important class of orthogonal systems includes of orthogonal polynomials $p_n(x)$, ($n = 0, 1, 2, \dots$). This class consists of degree n such as Legendre, Jacobi polynomials, Hermit, Laguerre, and Chebyshev polynomials. Moreover, the orthogonality property of these polynomials has many other general properties, the most important property of orthogonal polynomials is their orthogonality. In many mathematical models, orthogonal polynomials are used as a basis and this basis converts an IEs as a simple algebraic series of orthogonal polynomials. Additionally, one problem of a particular interest is the orthogonality property; actually orthogonality property is defined with respect to an inner product, if we have a linear space of functions, a function space $L^2_2[-1, 1]$ the square integrable real valued, functions on $[-1, 1]$, we can expand a function on $L^2_2[-1, 1]$ as a sum of these bases, in which the contributions of researcher mathematicians like Bernstein, Chebyshev, etc play a distinguished role see Mason & Handscomb (2002).

Let interval $[a, b] = [-1, 1]$ and weight functions be defined by

$$w_{(i,k)}(x) = \begin{cases} \frac{x^{k-1}}{\sqrt{1-x^{2k}}}, & i = 1, \\ x^{k-1} \sqrt{1-x^{2k}}, & i = 2. \end{cases} \quad (3.20)$$

where k is positive odd integers. Then the inner product of $Z^k_{(i,n)}(x)$ is defined as

$$\langle Z^k_{(i,n)}, Z^k_{(i,m)} \rangle_{w_{(i,k)}} = \begin{cases} \int_{-1}^1 w_{(1,k)}(x) Z^k_{(1,n)}(x) Z^k_{(1,m)}(x) dx, & i = 1, \\ \int_{-1}^1 w_{(2,k)}(x) Z^k_{(2,n)}(x) Z^k_{(2,m)}(x) dx, & i = 2, \end{cases} \quad (3.21)$$

where k is positive odd integers; $n, m = 0, 1, 2, 3, \dots$

However, for the purposes of proving the main results, we need to state two Lemmas are precisely related for the proof.

Lemma 3.3.1 *Berthold et al. (1992)*

$$\int_0^\pi \frac{\cos(n\theta)}{\cos(\theta) - \cos(\phi)} d\theta = \pi \frac{\sin(n\phi)}{\sin(\phi)}, \quad (3.22)$$

for any ϕ in $[0, \pi]$, $n = 1, 2, 3, \dots$.

Lemma 3.3.2 *Berthold et al. (1992)*

$$\int_0^\pi \frac{\sin(n\theta) \sin(\theta)}{\cos(\theta) - \cos(\phi)} d\theta = -\pi \cos(n\phi), \quad (3.23)$$

for any ϕ in $[0, \pi]$, $n = 1, 2, 3, \dots$.

The main properties ECPs of the first and second kinds of order k are summarized in the following Lemma:

Lemma 3.3.3 *Let $Z_{(i,n)}^k(x)$, $i = \{1, 2\}$ be the set of Extended Chebyshev polynomials of the first and second kinds of order k , respectively and consider that the weight functions $w_{(i,k)}$, $i = \{1, 2\}$ defined in Eq. (3.20). Then:*

(i) The three terms recurrence relations between ECPs of the first and second kinds :

$$Z_{(1,n)}^k(x) = 2x^k Z_{(1,n-1)}^k(x) - Z_{(1,n-2)}^k(x), \quad (3.24)$$

$$Z_{(2,n)}^k(x) = 2x^k Z_{(2,n-1)}^k(x) - Z_{(2,n-2)}^k(x). \quad (3.25)$$

(ii) The relationship between ECPs of the first and second kinds:

$$Z_{(2,n)}^k(x) - Z_{(2,n-2)}^k(x) = 2Z_{(1,n)}^k(x) \quad (3.26)$$

(iii) Orthogonality property of ECPs of the first kind of order k :

$$\langle Z_{(1,n)}^k, Z_{(1,m)}^k \rangle_{w(1,k)} = \begin{cases} 0, & m \neq n, \\ \frac{\pi}{2k}, & m = n, \\ \frac{\pi}{k}, & m = n = 0. \end{cases} \quad (3.27)$$

where $\langle . \rangle$ and $w_{(1,k)}(x)$ defined in Eq. (3.21) and (3.20) respectively.

(iv) Orthogonality property of ECPs of the second kind:

$$\langle Z_{(2,n)}^k, Z_{(2,m)}^k \rangle_{w(2,k)} = \begin{cases} 0, & m \neq n, \\ \frac{\pi}{2k}, & m = n. \end{cases} \quad (3.28)$$

where $\langle . \rangle$ and $w_{(2,k)}(x)$ defined in Eq. (3.21) and (3.20) respectively.

(v) For each $t \in (-1, 1)$

$$\int_{-1}^1 w_{(1,k)}(x) \frac{Z_{(1,n)}^k(x)}{x^k - t^k} dx = \frac{\pi}{k} Z_{(2,n-1)}^k(t), \quad n = 1, 2, \dots; \quad (3.29)$$

(vi) For each $t \in (-1, 1)$

$$\int_{-1}^1 w_{(2,k)}(x) \frac{Z_{(2,n-1)}^k(t)}{x^k - t^k} dx = -\frac{\pi}{k} Z_{(1,n)}^k(t), \quad n = 1, 2, \dots; \quad (3.30)$$

Proof:

Even if equality Lemma 3.3.1 is a known proof of it, it is not given in detail. Here, we prove the Lemma 3.3.1 based on complex trigonometric functions and Residue Theorem, therefore we sketch the proof in detail. First we want to show that:

$$I_n(\phi) = \int_0^\pi \frac{\cos(n\theta)}{\cos(\theta) - \cos(\phi)} d\theta = \frac{1}{2} \int_{-\pi}^\pi \frac{\cos(n\theta)}{\cos(\theta) - \cos(\phi)} d\theta, \quad \phi \in [0, \pi]. \quad (3.31)$$

Now;

$$\frac{1}{2} \int_{-\pi}^\pi \frac{\cos(n\theta)}{\cos(\theta) - \cos(\phi)} d\theta = \frac{1}{2} \int_{-\pi}^0 \frac{\cos(n\theta)}{\cos(\theta) - \cos(\phi)} d\theta + \frac{1}{2} \int_0^\pi \frac{\cos(n\theta)}{\cos(\theta) - \cos(\phi)} d\theta \quad (3.32)$$

For the first right hand side of Eq. (3.32), let $u = -\theta \Rightarrow -du = d\theta$, when $\theta = -\pi, u = \pi$, also when $\theta = 0, u = 0$, we get:

$$\begin{aligned} \frac{1}{2} \int_{-\pi}^\pi \frac{\cos(n\theta)}{\cos(\theta) - \cos(\phi)} d\theta &= \frac{1}{2} \int_{-\pi}^0 \frac{\cos(n\theta)}{\cos(\theta) - \cos(\phi)} d\theta + \frac{1}{2} \int_0^\pi \frac{\cos(n\theta)}{\cos(\theta) - \cos(\phi)} d\theta \\ &= \frac{1}{2} \int_\pi^0 \frac{\cos(-nu)}{\cos(-u) - \cos(\phi)} du + \frac{1}{2} \int_0^\pi \frac{\cos(n\theta)}{\cos(\theta) - \cos(\phi)} d\theta \\ &= \frac{1}{2} \int_0^\pi \frac{\cos(n\theta)}{\cos(\theta) - \cos(\phi)} d\theta + \frac{1}{2} \int_0^\pi \frac{\cos(n\theta)}{\cos(\theta) - \cos(\phi)} d\theta \\ &= \int_0^\pi \frac{\cos(n\theta)}{\cos(\theta) - \cos(\phi)} d\theta \end{aligned}$$

Since the cosine function is an even function and u is a dummy variable, switching bounds of definite integral.

It is known that the $\cos(\theta)$ related with a complex variable z in the form:

$$\cos(\theta) = \frac{1}{2} \left(z + \frac{1}{z} \right), \quad (3.33)$$

where $z = re^{i\theta}$ with $|z| = r = 1$.

If z moves on the unit circle, centered at the origin, $\cos(\theta)$ traverses the interval $[-1, 1]$ twice. Using Euler formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta), \quad (3.34)$$

we obtain

$$z = re^{i\theta} \Rightarrow d\theta = \frac{dz}{iz} \quad (3.35)$$

$$\cos(n\theta) = \frac{e^{in\theta} + e^{-in\theta}}{2} = \frac{z^n + \frac{1}{z^n}}{2}. \quad (3.36)$$

Substituting Eq. (3.33)-(3.36) into Eq. (3.31) and note that the quadratic term $z^2 - 2\cos(\phi)z + 1$ has roots $z = \cos(\phi) \pm \sqrt{\cos^2(\phi) - 1} \Rightarrow z_1 = e^{i\phi}, z_2 = e^{-i\phi}$ yields:

$$\begin{aligned} I_n(\phi) &= \frac{1}{2} \oint_{|z|=1} \frac{\frac{1}{2} \left(z^n + \frac{1}{z^n} \right)}{\frac{1}{2} \left(z + \frac{1}{z} \right) - \cos(\phi)} \frac{dz}{iz} = \frac{1}{2i} \oint_{|z|=1} \frac{z^{2n} + 1}{z^n (z^2 - 2\cos(\phi)z + 1)} dz \\ &= \frac{1}{2i} \oint_{|z|=1} \frac{z^n}{(z - e^{i\phi})(z - e^{-i\phi})} dz + \frac{1}{2i} \oint_{|z|=1} \frac{1}{z^n (z - e^{i\phi})(z - e^{-i\phi})} dz \\ &= I_{1n}(\phi) + I_{2n}(\phi), \end{aligned} \quad (3.37)$$

where,

$$I_{1n}(\phi) = \frac{1}{2i} \oint_{|z|=1} \frac{z^n}{(z - e^{i\phi})(z - e^{-i\phi})} dz \quad (3.38)$$

$$I_{2n}(\phi) = \frac{1}{2i} \oint_{|z|=1} \frac{1}{z^n (z - e^{i\phi})(z - e^{-i\phi})} dz \quad (3.39)$$

Thus, $I_n(\phi)$ in (3.37) is divided into two parts.

- For the first part $I_{1n}(\phi)$ of right hand side of Eq. (3.38), the function

$$g(z) = \frac{z^n}{(z - e^{i\phi})(z - e^{-i\phi})}, \quad (3.40)$$

has two simple poles of order one ($z_1 = e^{i\phi}$, and $z_2 = e^{-i\phi}$).

Using Residue Theorem (Theorem 2.3.2 at the page 17), The residue for $z_1 = e^{i\phi}$

$$\begin{aligned} \text{Res}(g(z), z_1 = e^{i\phi}) &= \lim_{z \rightarrow e^{i\phi}} \left[(z - e^{i\phi}) \frac{z^n}{(z - e^{i\phi})(z - e^{-i\phi})} \right] \\ &= \frac{e^{in\phi}}{(e^{i\phi} - e^{-i\phi})} \end{aligned} \quad (3.41)$$

We obtain:

$$\text{Res}(g(z), z_1 = e^{i\phi}) = \frac{e^{in\phi}}{e^{i\phi} - e^{-i\phi}}, \quad (3.42)$$

$$\text{Res}(g(z), z_2 = e^{-i\phi}) = -\frac{e^{-in\phi}}{e^{i\phi} - e^{-i\phi}}. \quad (3.43)$$

Substitute Eq. (3.42) and (3.43) into (3.38) to get:

$$\begin{aligned} I_{1n}(\phi) &= \frac{1}{2i} \oint_{|z|=1} \frac{z^n}{(z - e^{i\phi})(z - e^{-i\phi})} dz \\ &= \frac{1}{2i} 2\pi i [\text{Res}(g(z), z_1) + \text{Res}(g(z), z_2)] \\ &= \pi \left[\frac{\frac{e^{in\phi} - e^{-in\phi}}{2i}}{\frac{e^{i\phi} - e^{-i\phi}}{2i}} \right] \\ &= \pi \frac{\sin(n\phi)}{\sin(\phi)} \end{aligned} \quad (3.44)$$

- For the second part $I_{2n}(\phi)$ of Eq. (3.37), the function

$$f(z) = \frac{1}{z^n(z - e^{i\phi})(z - e^{-i\phi})}, \quad (3.45)$$

has three poles, two of them are of order one ($z_1 = e^{i\phi}$, $z_2 = e^{-i\phi}$), and the other pole is at $z = 0$ with order n , the residue of $f(z)$ at $z_1 = e^{i\phi}$ and $z_2 = e^{-i\phi}$ are

$$\text{Res}(f(z), z_1 = e^{i\phi}) = \frac{e^{in\phi}}{(e^{i\phi} - e^{-i\phi})}, \quad (3.46)$$

$$\text{Res}(f(z), z_2 = e^{-i\phi}) = \frac{e^{-in\phi}}{(e^{-i\phi} - e^{i\phi})}. \quad (3.47)$$

Applying Theorem 2.3.2 at the page 17, we find that the residue of $f(z)$ at $z_3 = 0$, with order n is:

$$\begin{aligned}
Res(f(z), z_3 = 0) &= \lim_{z \rightarrow 0} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left[(z-0)^n \frac{1}{z^n(z-e^{i\phi})(z-e^{-i\phi})} \right] \\
&= \frac{1}{(n-1)!} \frac{1}{2i \sin(\phi)} \lim_{z \rightarrow 0} \frac{d^{n-1}}{dz^{n-1}} \left[\frac{1}{(z-e^{i\phi})} - \frac{1}{(z-e^{-i\phi})} \right] \\
&= \frac{1}{(n-1)!} \frac{1}{2i \sin(\phi)} \left[\frac{(-1)^{n-1}(n-1)!}{(-1)^n e^{in\phi}} - \frac{(-1)^{n-1}(n-1)!}{(-1)^n e^{-in\phi}} \right] \\
&= \frac{1}{2i \sin(\phi)} \left[\frac{-1}{e^{in\phi}} + \frac{1}{e^{-in\phi}} \right] \\
&= \frac{1}{2i \sin(\phi)} \left[e^{in\phi} - e^{-in\phi} \right] \\
&= \frac{1}{\sin(\phi)} \left[\frac{e^{in\phi} - e^{-in\phi}}{2i} \right] \\
&= \frac{\sin(n\phi)}{\sin(\phi)}
\end{aligned} \tag{3.48}$$

It should be noted that the term can be written as a partial fraction:

$$\frac{1}{(z-e^{i\phi})(z-e^{-i\phi})} = \frac{a}{(z-e^{i\phi})} + \frac{b}{(z-e^{-i\phi})}$$

where is $a = -b = \frac{1}{2i \sin(\phi)}$. Substituting Eqs. (3.46)-(3.48) into (3.39) yields:

$$\begin{aligned}
I_{2n}(\phi) &= \frac{1}{2i} 2\pi i \left[Res(f(z), z_1 = e^{i\phi}) + Res(f(z), z_1 = e^{-i\phi}) + Res(f(z), z_0 = 0) \right] \\
&= \pi \left[\frac{e^{-in\phi}}{e^{in\phi} - e^{-in\phi}} + \frac{e^{in\phi}}{e^{-in\phi} - e^{in\phi}} + \frac{\sin n\phi}{\sin \phi} \right] \\
&= \pi \left[\frac{e^{-in\phi}}{e^{in\phi} - e^{-in\phi}} - \frac{e^{in\phi}}{e^{in\phi} - e^{-in\phi}} + \frac{\sin n\phi}{\sin \phi} \right] \\
&= \pi \left[\frac{e^{-in\phi} - e^{in\phi}}{e^{in\phi} - e^{-in\phi}} + \frac{\sin n\phi}{\sin \phi} \right] \\
&= \pi \left[\frac{e^{in\phi} - e^{-in\phi}}{e^{in\phi} - e^{-in\phi}} + \frac{\sin n\phi}{\sin \phi} \right] \\
&= \pi \left[\frac{\sin n\phi}{\sin \phi} + \frac{\sin n\phi}{\sin \phi} \right] \\
&= 2\pi \frac{\sin n\phi}{\sin \phi}
\end{aligned} \tag{3.49}$$

The Proof of Lemma 3.3.1 can be completed by substituting (3.44) and (3.49) into (3.37).

Proof of Lemma 3.3.2 using trigonometric identity $\sin(n\theta) \sin(\theta) = \frac{1}{2} [\cos((n-1)\theta) -$

$\cos((n+1)\theta)]$ and applying Lemma 3.3.1 yields:

$$\begin{aligned}
 \int_0^\pi \frac{\sin(n\theta) \sin(\theta)}{\cos(\theta) - \cos(\phi)} d\theta &= \frac{1}{2} \int_0^\pi \frac{\cos((n-1)\theta)}{\cos(\theta) - \cos(\phi)} d\theta - \frac{1}{2} \int_0^\pi \frac{\cos(n+1)\theta}{\cos(\theta) - \cos(\phi)} d\theta \\
 &= \frac{1}{2} \pi \left[\frac{\sin((n-1)\phi)}{\sin(\phi)} - \frac{1}{2} \pi \frac{\sin((n+1)\phi)}{\sin(\phi)} \right] \\
 &= \pi \left[\frac{\cos(\frac{(n-1)\phi + (n+1)\phi}{2}) \sin(\frac{(n-1)\phi - (n+1)\phi}{2})}{\sin(\phi)} \right] \\
 &= \pi \frac{\cos(n\phi) \sin(-\phi)}{\sin(\phi)} \\
 &= -\pi \cos(n\phi)
 \end{aligned} \tag{3.50}$$

Proof of Lemma 3.3.3:

- (i) Using the trigonometric identity $2\cos(\theta) \sin(n\theta) = \sin(n+1)\theta - \sin(n-1)\theta$, divided both sides by $\sin(\theta)$ we obtain the results.
- (ii) Applying the trigonometric identity $2\sin(\theta) \cos(n\theta) = \sin(n+1)\theta - \sin(n-1)\theta$, divided both side by $\sin(\theta)$, leads us to a relationship between $Z_{(1,n)}^k(x)$ and $Z_{(2,n)}^k(x)$.
- (iii) Using the definitions of $Z_{(1,n)}(x) = \cos(n \cos^{-1}(x^k))$ and $Z_{(1,m)}(x) = \cos(m \cos^{-1}(x^k))$, for $n, m = \{0, 1, 2, 3, \dots\}$ and letting $u = x^k \Rightarrow du = kx^{k-1} dx$, we obtain,

$$\begin{aligned}
 \langle Z_{(1,n)}, Z_{(1,m)} \rangle_{w(1,k)} &= \int_{-1}^1 \frac{x^{k-1} Z_{(1,n)}(x) Z_{(1,m)}(x)}{\sqrt{1-x^{2k}}} dx \\
 &= \frac{1}{k} \int_0^\pi \frac{\cos(n\theta) \cos(m\theta) \sin(\theta)}{\sqrt{1-\cos^2(\theta)}} d\theta \\
 &= \frac{1}{k} \int_0^\pi \cos(n\theta) \cos(m\theta) d\theta \\
 &= 0, n \neq m.
 \end{aligned} \tag{3.51}$$

Hence

$$\langle Z_{(1,n)}, Z_{(1,m)} \rangle = 0, (n \neq m) \tag{3.52}$$

The norm of $Z_{(1,n)}(x)$ is

$$\begin{aligned}
 \|Z_{(1,n)}\|^2 &= \int_{-1}^1 \frac{x^{k-1} [Z_{(1,n)}(x)]^2}{\sqrt{1-x^{2k}}} dx \\
 &= \frac{1}{k} \int_0^\pi \frac{\cos(n\theta) \cos(n\theta) \sin(\theta)}{\sqrt{1-\sin^2(\theta)}} d\theta \\
 &= \frac{1}{2k} \left[\left(\theta + \frac{\sin(2n\theta)}{2n} \right) \right]_0^\pi, n \neq 0 \\
 &= \frac{\pi}{2k}, n = 1, 2, \dots
 \end{aligned} \tag{3.53}$$

while, $n = 0$

$$\|Z_{(1,0)}\|^2 = \frac{\pi}{k}. \tag{3.54}$$

(iv) Using the relations $Z_{(2,n)}^k(x)$ and $Z_{(2,n)}^k(x)$, for $n, m = \{0, 1, 2, 3, \dots\}$ and letting $u = x^k \Rightarrow du = kx^{k-1} dx$, we get,

$$\begin{aligned}
 \langle Z_{(2,n)}^k, Z_{(2,m)}^k \rangle_{w(2,k)} &= \int_{-1}^1 w_{(2,k)}(x) Z_{(2,n)}^k Z_{(2,m)}^k dx \\
 &= \int_{-1}^1 x^{k-1} \sqrt{1-x^{2k}} \frac{\sin((n+1)\cos^{-1}(x^k))}{\sin(\cos^{-1}(x^k))} \frac{\sin((m+1)\cos^{-1}(x^k))}{\sin(\cos^{-1}(x^k))} dx \\
 &= \frac{1}{k} \int_{-1}^1 \sqrt{1-u^2} \frac{\sin((n+1)\cos^{-1}(u))}{\sin(\cos^{-1}(u))} \frac{\sin((m+1)\cos^{-1}(u))}{\sin(\cos^{-1}(u))} du \\
 &= \frac{1}{k} \int_0^\pi \sin(n+1)\theta \sin(m+1)\theta d\theta \\
 &= \frac{1}{2k} \int_0^\pi [\cos(n-m)\theta - \cos(n+m+2)\theta] d\theta \\
 &= \frac{1}{2k} \left[\frac{\sin(n-m)\theta}{(n-m)} - \frac{\sin(n+m+2)\theta}{(n+m+2)} \right]_0^\pi, n \neq m \\
 &= 0
 \end{aligned}$$

Hence, the norm of $Z_{(2,n)}^k(x)$ is

$$\begin{aligned}
 \|Z_{(2,n)}^k\|^2 &= \frac{1}{k} \int_0^\pi \sin^2(n+1)\theta d\theta \\
 &= \frac{\pi}{2k}
 \end{aligned} \tag{3.55}$$

(v) Using the definition of $Z_{(1,n)}^k(x)$ and applying Lemma 3.3.1 together with replacing $w_{(1,k)}(x) = \frac{x^{k-1}}{\sqrt{1-x^{2k}}}$ and letting $u = x^k \Rightarrow du = kx^{k-1} dx$ with $x = -1 \Rightarrow u = -1$, and

$x = 1 \Rightarrow u = 1$, we obtain

$$\begin{aligned}
 \int_{-1}^1 w_{(1,k)}(x) \frac{Z_{(1,n)}^k(x)}{x^k - t^k} dx &= \frac{1}{k} \int_{-1}^1 \frac{x^{k-1}}{x^k - t^k} \frac{\cos(n \cos^{-1}(x^k))}{\sqrt{1 - x^{2k}}} dx \\
 &= \frac{1}{k} \int_{-1}^1 \frac{1}{u - t^k} \frac{\cos(n \cos^{-1}(u))}{\sqrt{1 - u^2}} du \\
 &= \frac{1}{k} \int_0^\pi \frac{1}{\cos(\theta) - \cos(\phi)} \frac{\cos(n\theta)}{\sqrt{1 - \cos^2(n\theta)}} \sin(\theta) d\theta, \quad \phi(t) = t^k \\
 &= \frac{1}{k} \int_0^\pi \frac{\cos(n\theta)}{\cos(\theta) - \cos(\phi)} d\theta \\
 &= \frac{\pi \sin((n-1) + 1)\phi}{k \sin(\phi)} \\
 &= \frac{\pi}{k} Z_{(2,n-1)}^k(t)
 \end{aligned}$$

where $n = 1, 2, 3, \dots$; k is positive odd integer, when $t \in (-1, 1) \Rightarrow \phi(t) = t^k \in (-1, 1)$ since k is positive odd integer, so our assumption is well defined.

(vi) Substituting $Z_{(2,n)}^k(x)$ and applying Lemma 3.3.2 together with replacing $w_{(2,k)}(x) = x^{k-1} \sqrt{1 - x^{2k}}$ and letting $u = x^k \Rightarrow du = kx^{k-1} dx$ with $x = -1 \Rightarrow u = -1$, and $x = 1 \Rightarrow u = 1$, we have

$$\begin{aligned}
 \int_{-1}^1 w_{(2,k)}(x) \frac{Z_{(2,n-1)}^k(t)}{x^k - t^k} dx &= \int_{-1}^1 \frac{x^{k-1} \sqrt{1 - x^{2k}} \sin(n \cos^{-1} x^k)}{(x^k - t^k) \sin(\cos^{-1} x^k)} dx \\
 &= \frac{1}{k} \int_{-1}^1 \frac{\sqrt{1 - u^2} \sin(n \cos^{-1} u)}{(u - t^k) \sin(\cos^{-1} u)} du \\
 &= -\frac{1}{k} \int_0^\pi \frac{\sin(n\theta) \sin(\theta)}{(\cos(\theta) - \cos(\phi))} d\theta, \quad \phi(t) = t^k \\
 &= -\frac{\pi}{k} Z_{(1,n)}^k(t)
 \end{aligned}$$

where $n = 1, 2, 3, \dots$; k is positive odd integer.

Theorem 3.3.4 Berthold et al. (1992). The integral equation

$$\lambda \phi(t) = \int_{-1}^1 \frac{\phi(x) \log|x - t|}{\sqrt{1 - x^2}} dx$$

has eigenfunctions $\phi(t) = \phi_n(t) = T_n(t)$ corresponding eigenvalues $\lambda = \lambda_n = \frac{\pi}{n}$, $n = 1, 2, 3, \dots$ where $T_n(t)$ is Chebyshev polynomials of the first kind.

Proof: Proof of the Theorem 3.3.4 can be found in Berthold et al. (1992).

Using the definition of Chebyshev polynomials of the first kind $T_n(x) = \cos(n \cos^{-1}(x))$ and replacing $\phi(x)$ by $T_n(x)$ and letting $x = \cos(\theta) \Rightarrow dx = -\sin(\theta) d\theta$, we get:

$$\begin{aligned}
I_0^*(t) &= \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} \phi(x) \log|x-t| dx = \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} T_n(x) \log|x-t| dx, \quad \text{letting } t = \cos(\psi) \\
&= \int_0^\pi \frac{\sin(\theta)}{\sqrt{1-\cos^2(\theta)}} \cos(n\theta) \log|\cos(\theta) - \cos(\psi)| d\theta, \\
&= \int_0^\pi \frac{\sin(\theta)}{\sin(\theta)} \cos(n\theta) \log|\cos(\theta) - \cos(\psi)| d\theta, \\
&= \int_0^\pi \cos(n\theta) \log|\cos(\theta) - \cos(\psi)| d\theta
\end{aligned} \tag{3.56}$$

Consider $dv = \cos(n\theta)d\theta \Rightarrow v = \frac{1}{n} \sin(n\theta)$, $u = \log|\cos(\theta) - \cos(\psi)| \Rightarrow du = -\frac{\sin(\theta)}{\cos(\theta) - \cos(\psi)} d\theta$, then using integration by parts, Eq. (3.56) yields

$$\begin{aligned}
I_0^*(t) &= \frac{1}{n} \sin(n\theta) \log|\cos(\theta) - \cos(\psi)| \Big|_0^\pi + \frac{1}{n} \int_0^\pi \frac{\sin(n\theta) \sin(\theta)}{\cos(\theta) - \cos(\psi)} d\theta \\
&= \frac{1}{n} \int_0^\pi \frac{\sin(n\theta) \sin(\theta)}{\cos(\theta) - \cos(\psi)} d\theta, \quad \text{applying Lemma 3.3.2} \\
&= -\frac{\pi}{n} \cos(n\psi) \\
&= -\frac{\pi}{n} T_n(t)
\end{aligned}$$

Now, we use the Lemma 3.3.1 and Theorem 3.3.3 to prove next Theorem 3.3.5 which is related to Extended Chebyshev polynomials of the first kind $Z_{(1,n)}^k(x)$.

Theorem 3.3.5 *The integral equation*

$$\lambda \phi(t) = \int_{-1}^1 \frac{x^k - 1}{\sqrt{1-x^2}} \phi(x) \log|x^k - t^k| dx,$$

has eigenfunctions $\phi(t) = \phi_n(t) = Z_{(1,n)}^k(t)$ corresponding eigenvalues $\lambda = \lambda_n = -\frac{\pi}{nk}$, $n = 1, 2, 3, \dots$ where $Z_{(1,n)}^k(t)$ is the Extended Chebyshev polynomials of the first kind of order k , k is positive odd integers.

Proof: Using the definition ECPs of the first kind, $Z_{(1,n)}^k(x) = \cos(n\theta) = \cos(n \cos^{-1}(x^k))$, $x^k = \cos(\theta)$, and applying Theorem 3.3.4 together with replacing $\phi(x) = Z_{(1,n)}^k(x) = \cos(n \cos^{-1}(x^k))$ and letting $u = x^k \Rightarrow du = kx^{k-1} dx$ with $x = -1 \Rightarrow u = -1$, and $x = 1 \Rightarrow u = 1$, we obtain:

$$\begin{aligned}
I_n(\phi) &= \int_{-1}^1 \frac{x^{k-1}}{\sqrt{1-x^{2k}}} Z_{(1,n)}^k(x) \log |x^k - t^k| dx \\
&= \int_{-1}^1 \frac{x^{k-1}}{\sqrt{1-x^{2k}}} \cos(n \cos^{-1}(x^k)) \log |x^k - t^k| dx \\
&= \frac{1}{k} \int_{-1}^1 \frac{\cos n(\cos^{-1}(u))}{\sqrt{1-u^2}} \log |u - t^k| du \\
&= \frac{1}{k} \int_{-1}^1 \frac{T_n(u)}{\sqrt{1-u^2}} \log |u - t^*| du, t^* = t^k \\
&= -\frac{\pi}{nk} T_n(t^*) \\
&= -\frac{\pi}{nk} T_n(t^k) \\
&= -\frac{\pi}{nk} Z_{(1,n)}^k(t).
\end{aligned}$$

where $n = 1, 2, 3, \dots; k$ is positive odd integers, (since $t \in (-1, 1) \rightarrow t^k \in (-1, 1)$).

For $n = 0$, first: it should be noted that:

$$\int_0^{\frac{\pi}{2}} \log \cos(\theta) d\theta = \int_0^{\frac{\pi}{2}} \log \sin(\theta) d\theta \quad (3.57)$$

Eq. (3.57) is obtained by letting $\theta = \frac{\pi}{2} - x$.

Second: we need to show that:

$$\int_0^{\pi} \log \sin(x) dx = -\pi \log 2 \quad (3.58)$$

To do that consider the integration:

$$\begin{aligned}
 I_1 &= \int_0^\pi \log \sin(x) dx = \int_0^{\frac{\pi}{2}} \log \sin(x) dx + \int_{\frac{\pi}{2}}^\pi \log \sin(x) dx, \quad \text{for the second part, let } x = \frac{\pi}{2} + y \\
 &= \int_0^{\frac{\pi}{2}} \log \sin(x) dx + \int_0^{\frac{\pi}{2}} \log \sin\left(\frac{\pi}{2} + y\right) dy \\
 &= \int_0^{\frac{\pi}{2}} \log \sin(x) dx + \int_0^{\frac{\pi}{2}} \log \cos(x) dx, \quad y \text{ is dummy variable} \\
 &= \int_0^{\frac{\pi}{2}} \log\left[\frac{1}{2} 2 \sin(x) \cos(x)\right] dx \\
 &= - \int_0^{\frac{\pi}{2}} \log 2 + \int_0^{\frac{\pi}{2}} \log 2 \sin(x) \cos(x) dx \\
 &= - \int_0^{\frac{\pi}{2}} \log 2 + \int_0^{\frac{\pi}{2}} \ln \sin(2x) dx \quad \text{let } 2x = y \\
 &= - \frac{\pi}{2} \log 2 + \frac{1}{2} \int_0^\pi \log \sin(y) dy \\
 &= - \frac{\pi}{2} \log 2 + \frac{1}{2} I_1 \\
 &= -\pi \log 2
 \end{aligned} \tag{3.59}$$

Now, we need to prove:

$$\int_{-1}^1 \frac{x^{k-1}}{\sqrt{1-x^{2k}}} Z_{(1,0)}^k(x) \log|x^k - t^k| dx = -\frac{\pi}{k} \log 2 \tag{3.60}$$

Using the definition of $Z_{(1,0)}^k(x) = 1$ and letting $u = x^k \Rightarrow du = kx^{k-1} dx$, we get,

$$\begin{aligned}
 I_0(\phi) &= \int_{-1}^1 \frac{x^{k-1}}{\sqrt{1-x^{2k}}} Z_{(1,0)}^k(x) \log|x^k - t^k| dx \\
 &= \int_{-1}^1 \frac{x^{k-1}}{\sqrt{1-x^{2k}}} \log|x^k - t^k| dx \\
 &= \frac{1}{k} \int_{-1}^1 \frac{1}{\sqrt{1-u^2}} \log|u - t^k| du, \quad \text{let } u = \cos \theta \text{ and } t^k = \cos(\phi) \\
 &= \frac{1}{k} \int_{-1}^1 \frac{1}{\sqrt{1-u^2}} \log|\cos \theta - \cos(\phi)| du, \\
 &= \frac{1}{k} \int_\pi^0 \frac{1}{\sqrt{1-\cos^2 \theta}} \log|\cos \theta - \cos(\phi)| \sin \theta d\theta, \\
 &= \frac{1}{k} \int_0^\pi \log|\cos \theta - \cos(\phi)| d\theta,
 \end{aligned} \tag{3.61}$$

where k is positive odd integers. (since $t \in (-1, 1)$, $\Rightarrow t^k \in (-1, 1)$, so $\cos(\phi)$ is well defined).

Differentiating $I_0(\phi)$ with respect to ϕ yields

$$\begin{aligned}
 \frac{d}{d\phi} I_0(\phi) &= I'_0(\phi) = \frac{1}{k} \frac{d}{d\phi} \int_0^\pi \log |\cos \theta - \cos(\phi)| d\theta \\
 &= \frac{1}{k} \int_0^\pi \frac{\sin(\phi)}{\cos \theta - \cos(\phi)} d\theta \\
 &= 0, \quad \text{apply Lemma 3.3.1 when } n=0
 \end{aligned} \tag{3.62}$$

Hence $I_0(\phi)$ must be a constant function, so assume $\cos(\phi) = 0$, we get:

$$\begin{aligned}
 I_0(0) &= \frac{1}{k} \int_0^\pi \log |\cos \theta| d\theta \\
 &= \frac{1}{k} \int_0^{\frac{\pi}{2}} \log |\cos \theta| d\theta + \frac{1}{k} \int_{\frac{\pi}{2}}^\pi \log |\cos \theta| d\theta, \quad \text{for 2nd part let } x = \theta - \frac{\pi}{2} \\
 &= \frac{1}{k} \int_0^{\frac{\pi}{2}} \log |\cos \theta| d\theta + \frac{1}{k} \int_0^{\frac{\pi}{2}} \log |\cos(x + \frac{\pi}{2})| dx \\
 &= \frac{1}{k} \int_0^{\frac{\pi}{2}} \log |\cos \theta| d\theta + \frac{1}{k} \int_0^{\frac{\pi}{2}} \log |\sin(\theta)| d\theta \\
 &= \frac{1}{k} \int_0^{\frac{\pi}{2}} \log \cos \theta d\theta + \frac{1}{k} \int_0^{\frac{\pi}{2}} \log \sin(\theta) d\theta, \quad \text{use Eq. (3.57) to get} \\
 &= \frac{1}{k} \int_0^{\frac{\pi}{2}} \log \sin(\theta) d\theta + \frac{1}{k} \int_0^{\frac{\pi}{2}} \log \sin(\theta) d\theta \\
 &= \frac{2}{k} \int_0^{\frac{\pi}{2}} \log \sin(\theta) d\theta \\
 &= \frac{1}{k} \int_0^\pi \log \sin(\theta) d\theta \quad \text{use Eq. (3.59) yield} \\
 &= -\frac{\pi}{k} \log 2
 \end{aligned} \tag{3.63}$$

It implies the Proof of the Theorem 3.3.5.

3.4 Function Approximation Via Extended Chebyshev Polynomials of The First and Second Kinds of Order k

Many methods of expanding functions in infinite series have been studied by researchers. In fact, the familiar Fourier series, Laurent series and Taylor series can all be used as orthogonal functions on appropriately chosen weights and domains.

In the present section, we link the spectral properties together in the context of expansions in mainly of the first and second kinds.

3.4.1 The Special Logarithmic Singular Integral Equations of Order k Via the First Kind Extended Chebyshev Polynomial of Order k

In this section, we will investigate homogeneous and inhomogeneous logarithmic singular integral equations of the first and second kinds (LogSIEk). The main technique is dealing with logarithmic kernels and reduce the IE problem into linear algebraic equations involving matrices. The key point is to approximate the density function as truncated series ECP of the first kind of order k , $Z_{(1,n)}^k(t), n = 0, 1, 2, \dots$.

3.4.2 Special Logarithmic Singular Integral Equations of the First Kind of Order k

Let us consider the linear LogSIEk of the first kind

$$\int_{-1}^1 L(x, t) f(x) dx = g(t), \quad x, t \in [-1, 1], \quad (3.64)$$

where the right hand side $g(t)$ is given and

$$L(x, t) = \log |x^k - t^k|, k \text{ is positive odd integer.} \quad (3.65)$$

$f(x)$ is an unknown function which needs to be determined.

Proceeding in the usual way, we approximate $f(x)$ by

$$f_n(x) = a_0 Z_{(1,0)}^k(x) \frac{x^{k-1}}{\sqrt{1-x^{2k}}} + \frac{x^{k-1}}{\sqrt{1-x^{2k}}} \sum_{i=1}^n a_i Z_{(1,i)}^k(x), \quad k \text{ is positive odd integer.} \quad (3.66)$$

Substitute Eq. (3.66) into Eq. (3.64) to get

$$\begin{aligned} a_0 \int_{-1}^1 \frac{x^{k-1}}{\sqrt{1-x^{2k}}} L(x, t) Z_{(1,0)}^k(x) dx + \int_{-1}^1 \frac{x^{k-1}}{\sqrt{1-x^{2k}}} \sum_{i=1}^n a_i L(x, t) Z_{(1,i)}^k(x) dx &= g(t) \\ a_0 \int_{-1}^1 \frac{x^{k-1}}{\sqrt{1-x^{2k}}} L(x, t) Z_{(1,0)}^k(x) dx + \sum_{i=1}^n a_i \int_{-1}^1 \frac{t^{k-1}}{\sqrt{1-x^{2k}}} L(x, t) Z_{(1,i)}^k(x) dx &= g(t), \end{aligned} \quad (3.67)$$

where $L(x, t)$ is defined in Eq. 3.65. Applying Theorem (3.3.5) Eq. (3.67) yields:

$$-a_0 \frac{\pi}{k} \log 2 Z_{(1,0)}^k - \pi \sum_{i=1}^n a_i \frac{Z_{(1,i)}^k(t)}{i \times k} = g(t) \quad (3.68)$$

where the coefficients $a_i, i = 0, 1, 2, \dots, n$ are unknown to be determined.

To determine the unknown coefficients $a_i, i = 0, 1, 2, \dots, n$, we use the classical Galerkin method, we multiply both sides of Eq. (3.68) by $Z_{(1,j)}^k(t), j = 0, 1, 2, \dots, n$, then integrating it over the interval $[-1, 1]$ and using Lemma (3.3.3), Eq. (3.68) becomes

For $j = 0$

$$\begin{aligned} a_0 \frac{-\pi}{k} \log 2 \langle Z_{(1,0)}^k, Z_{(1,0)}^k \rangle_{w_{(1,k)}} + a_1 \frac{-\pi}{1 \times k} \langle Z_{(1,1)}^k, Z_{(1,0)}^k \rangle_{w_{(1,k)}} + \dots \\ + a_{n-1} \frac{-\pi}{(n-1) \times k} \langle Z_{(1,n-1)}^k, Z_{(1,0)}^k \rangle_{w_{(1,k)}} + a_n \frac{-\pi}{n \times k} \langle Z_{(1,n)}^k, Z_{(1,0)}^k \rangle_{w_{(1,k)}} \\ = \langle g, Z_{(1,0)}^k \rangle_{w_{(1,k)}} \end{aligned} \quad (3.69)$$

where $\langle Z_{(1,n)}^k, Z_{(1,m)}^k \rangle_{w_{(1,k)}}$ is defined in Eq. (3.21). Using orthogonal property of ECPs of the first kind of order k , $Z_{(1,n)}^k(x)$ with weight function $w_{(1,k)}(x) = \frac{x^{k-1}}{\sqrt{1-x^{2k}}}$, Eq. (3.69) can be simplified to:

$$\begin{aligned} a_0 \frac{-\pi^2}{k^2} \log 2 + 0 + \dots + 0 = \langle g, Z_{(1,0)}^k \rangle_{w_{(1,k)}} \\ a_0 = -\frac{k^2}{\pi^2 \log 2} \langle g, Z_{(1,0)}^k \rangle_{w_{(1,k)}} \end{aligned} \quad (3.70)$$

For $j = 1$

$$\begin{aligned} a_0 \frac{-\pi}{k} \log 2 \langle Z_{(1,0)}^k, Z_{(1,1)}^k \rangle_{w_{(1,k)}} + a_1 \frac{-\pi}{1 \times k} \langle Z_{(1,1)}^k, Z_{(1,1)}^k \rangle_{w_{(1,k)}} + \dots \\ + a_{n-1} \frac{-\pi}{(n-1) \times k} \langle Z_{(1,n-1)}^k, Z_{(1,1)}^k \rangle_{w_{(1,k)}} + a_n \frac{-\pi}{n \times k} \langle Z_{(1,n)}^k, Z_{(1,1)}^k \rangle_{w_{(1,k)}} \\ = \langle g, Z_{(1,1)}^k \rangle_{w_{(1,k)}} \end{aligned} \quad (3.71)$$

Eq. (3.71) can be simplified to:

$$\begin{aligned} 0 + a_1 \frac{-\pi^2}{2k^2} + 0 + \dots + 0 = \langle g, Z_{(1,1)}^k \rangle_{w_{(1,k)}} \\ a_1 = -\frac{2k^2}{\pi^2} \langle g, Z_{(1,1)}^k \rangle_{w_{(1,k)}} \end{aligned} \quad (3.72)$$

Proceeding in the same way, we get:

$$a_j = \begin{cases} -\frac{k^2}{\pi^2 \log 2} \langle g, Z_{(1,0)}^k \rangle_{w_{(1,k)}}, & j = 0, \\ -\frac{2k^2}{\pi^2} \langle g, Z_{(1,j)}^k \rangle_{w_{(1,k)}}, & j = 1, 2, \dots, n. \end{cases} \quad (3.73)$$

3.4.3 Special Logarithmic Singular Integral Equations of the Second Kind of Order k

3.4.3.1 Inhomogeneous Case

Consider inhomogeneous LogSIE $_k$ of the second kind in the form:

$$f(t) = g(t) + \int_{-1}^1 L(x, t) f(x) dx, \quad x, t \in [-1, 1], \quad (3.74)$$

where the right hand side the function $g(t)$ is given and the logarithmic kernel function $L(x, t)$ is defined in (3.65), $f(t)$ is the unknown function to be determined.

Substitute Eq. (3.66) into Eq. (3.74) to get

$$\begin{aligned} a_0 w_{(1,k)}(t) Z_{(1,0)}^k(t) + \sum_{i=1}^n a_i w_{(1,k)}(t) Z_{(1,i)}^k(t) &= g(t) + a_0 \langle L, Z_{(1,0)}^k \rangle_{w_{(1,k)}} + \sum_{i=1}^n a_i \langle L, Z_{(1,i)}^k \rangle_{w_{(1,k)}} \\ a_0 w_{(1,k)}(t) Z_{(1,0)}^k(t) + \sum_{i=1}^n a_i w_{(1,k)}(t) Z_{(1,i)}^k(t) &= g(t) + a_0 \langle L, Z_{(1,0)}^k \rangle_{w_{(1,k)}} + a_1 \langle L, Z_{(1,1)}^k \rangle_{w_{(1,k)}} \\ &\quad + \cdots + a_n \langle L, Z_{(1,n)}^k \rangle_{w_{(1,k)}} \end{aligned} \quad (3.75)$$

where $\langle \cdot, \cdot \rangle_{w_{(1,k)}}$ is defined in Eq. (3.21).

Applying Theorem (3.3.5) Eq. (3.75) becomes:

$$a_0 w_{(1,k)}(t) Z_{(1,0)}^k(t) + \sum_{i=1}^n a_i w_{(1,k)}(t) Z_{(1,i)}^k(t) = g(t) - a_0 \frac{\pi}{k} \log 2 - \pi \sum_{i=1}^n a_i \frac{Z_{(1,i)}^k(t)}{i \times k} \quad (3.76)$$

where the coefficients $a_i, i = 0, 1, 2, \dots, n$ are unknown to be determined.

We multiply both sides of Eq. (3.76) by $Z_{(1,j)}^k(t), j = 0, 1, 2, \dots, n$, then integrating it over the interval $[-1, 1]$ and using the orthogonality property (Lemma 3.3.3) of ECPs of the first kind $Z_{(1,j)}^k(x)$, Eq. (3.76) yields

$$\begin{aligned} a_0 \langle Z_{(1,0)}^k, Z_{(1,j)}^k \rangle_{w_{(1,k)}} + \sum_{i=1}^n a_i \langle Z_{(1,0)}^k, Z_{(1,j)}^k \rangle_{w_{(1,k)}} &= \langle g, Z_{(1,j)}^k \rangle + a_0 \frac{-\pi}{k} \log 2 \langle Z_{(1,0)}^k, Z_{(1,j)}^k \rangle + \cdots \\ &\quad + \sum_{i=1}^n \frac{-a_i \pi}{i \times k} \langle Z_{(1,i)}^k, Z_{(1,j)}^k \rangle \end{aligned} \quad (3.77)$$

where $j = 0, 1, 2, \dots, n$,

The linear simultaneous equations generated from Eq. (3.77) formulate a system of linear algebraic equations as:

For $j = 0$

$$\begin{aligned} a_0 \frac{\pi}{k} + 0 + \dots + 0 &= \langle g, Z_{(1,0)}^k \rangle + a_0 \frac{-\pi}{k} \log 2 \langle Z_{(1,0)}^k, Z_{(1,0)}^k \rangle + \dots + \sum_{i=1}^n \frac{-a_i \pi}{i \times k} \langle Z_{(1,i)}^k, Z_{(1,0)}^k \rangle \\ a_0 \frac{\pi}{k} [\log 2 \langle Z_{(1,0)}^k, Z_{(1,0)}^k \rangle + 1] &+ a_1 \frac{\pi}{1 \times k} \langle Z_{(1,1)}^k, Z_{(1,0)}^k \rangle + a_2 \frac{\pi}{2 \times k} \langle Z_{(1,2)}^k, Z_{(1,0)}^k \rangle + \dots \\ &+ a_n \frac{\pi}{n \times k} \langle Z_{(1,n)}^k, Z_{(1,0)}^k \rangle = \langle g, Z_{(1,0)}^k \rangle \end{aligned} \quad (3.78)$$

For $j = 1$

$$\begin{aligned} 0 + a_1 \frac{\pi}{2k} + 0 + \dots + 0 &= \langle g, Z_{(1,1)}^k \rangle + a_0 \frac{-\pi}{k} \log 2 \langle Z_{(1,0)}^k, Z_{(1,1)}^k \rangle + \dots + \sum_{i=1}^n \frac{-a_i \pi}{i \times k} \langle Z_{(1,i)}^k, Z_{(1,1)}^k \rangle \\ a_0 \frac{\pi}{k} \log 2 \langle Z_{(1,0)}^k, Z_{(1,1)}^k \rangle &+ a_1 \frac{\pi}{k} [\frac{1}{2} + \frac{1}{1} \langle Z_{(1,1)}^k, Z_{(1,1)}^k \rangle] + a_2 \frac{\pi}{2 \times k} \langle Z_{(1,2)}^k, Z_{(1,1)}^k \rangle + \dots \\ &+ a_n \frac{\pi}{n \times k} \langle Z_{(1,n)}^k, Z_{(1,1)}^k \rangle = \langle g, Z_{(1,1)}^k \rangle \end{aligned} \quad (3.79)$$

⋮

For $j = m$

$$\begin{aligned} 0 + \dots + a_m \frac{\pi}{2k} + \dots + 0 &= \langle g, Z_{(1,m)}^k \rangle + a_0 \frac{-\pi}{k} \log 2 \langle Z_{(1,0)}^k, Z_{(1,m)}^k \rangle + \dots + \sum_{i=1}^n \frac{-a_i \pi}{i \times k} \langle Z_{(1,i)}^k, Z_{(1,m)}^k \rangle \\ a_0 \frac{\pi}{k} \log 2 \langle Z_{(1,0)}^k, Z_{(1,m)}^k \rangle &+ \dots + a_m \frac{\pi}{k} [\frac{1}{2} + \frac{1}{m} \langle Z_{(1,m)}^k, Z_{(1,m)}^k \rangle] + \dots \\ &+ a_n \frac{\pi}{n \times k} \langle Z_{(1,n)}^k, Z_{(1,m)}^k \rangle = \langle g, Z_{(1,m)}^k \rangle \end{aligned} \quad (3.80)$$

⋮

For $j = n$

$$\begin{aligned}
 0 + 0 + \dots + a_n \frac{\pi}{2k} &= \langle g, Z_{(1,n)}^k \rangle + a_0 \frac{-\pi}{k} \log 2 \langle Z_{(1,0)}^k, Z_{(1,n)}^k \rangle + \dots + \sum_{i=1}^n \frac{-a_i \pi}{i \times k} \langle Z_{(1,i)}^k, Z_{(1,n)}^k \rangle \\
 a_0 \frac{\pi}{k} \log 2 \langle Z_{(1,0)}^k, Z_{(1,n)}^k \rangle &+ a_1 \frac{\pi}{1 \times k} \langle Z_{(1,1)}^k, Z_{(1,n)}^k \rangle + a_2 \frac{\pi}{2 \times k} \langle Z_{(1,2)}^k, Z_{(1,n)}^k \rangle + \dots \\
 &+ a_n \frac{\pi}{k} \left[\frac{1}{2} + \frac{1}{n} \langle Z_{(1,n)}^k, Z_{(1,n)}^k \rangle \right] = \langle g, Z_{(1,n)}^k \rangle
 \end{aligned} \quad (3.81)$$

The unknown coefficients $a_i, i = 0, 1, 2, 3, \dots, n$ can be determined by solving the system of linear equations which can be represented by a matrix form as:

$$\begin{bmatrix} \frac{\pi}{k} [\log 2 \beta_{(0,0)}^k + 1] & \pi \frac{\beta_{(0,1)}^k}{1 \times k} & \pi \frac{\beta_{(0,2)}^k}{2 \times k} & \pi \frac{\beta_{(0,3)}^k}{3 \times k} & \dots & \pi \frac{\beta_{(0,n)}^k}{n \times k} \\ \frac{\pi}{k} \log 2 \beta_{(1,0)}^k & \pi \left[\frac{1}{2 \times k} + \frac{\beta_{(1,1)}^k}{1 \times k} \right] & \pi \frac{\beta_{(1,2)}^k}{2 \times k} & \pi \frac{\beta_{(1,3)}^k}{3 \times k} & \dots & \pi \frac{\beta_{(1,n)}^k}{n \times k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\pi}{k} \log 2 \beta_{(m,0)}^k & \pi \frac{\beta_{(m,1)}^k}{1 \times k} & \pi \frac{\beta_{(m,2)}^k}{2 \times k} & \dots & \pi \left[\frac{1}{2 \times k} + \frac{\beta_{(m,m)}^k}{m \times k} \right] & \pi \frac{\beta_{(m,n)}^k}{n \times k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\pi}{k} \log 2 \beta_{(n,0)}^k & \pi \frac{\beta_{(n,1)}^k}{1 \times k} & \pi \frac{\beta_{(n,2)}^k}{2 \times k} & \pi \frac{\beta_{(n,3)}^k}{3 \times k} & \dots & \pi \left[\frac{1}{2 \times k} + \frac{\beta_{(n,n)}^k}{n \times k} \right] \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_m \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \langle g, Z_{(1,n)}^k \rangle \\ \vdots \\ \vdots \end{bmatrix} \quad (3.82)$$

where

$$\begin{aligned}
 \beta_{(m,j)}^k &= \langle Z_{(1,m)}^k, Z_{(1,j)}^k \rangle = \int_{-1}^1 Z_{(1,m)}^k(t) Z_{(1,j)}^k(t) dt \\
 C_j^k &= \langle g, Z_{(1,j)}^k \rangle
 \end{aligned} \quad (3.83)$$

$$m = 0, 1, 2, 3, \dots, n; j = 0, 1, 2, 3, \dots, n.$$

The above matrix has size $(n+1) \times (n+1)$, we use a software MATLAB program to determine coefficients $a_i, i = 0, 1, 2, 3, \dots, n$.

3.4.3.2 Homogeneous Case

For a homogeneous case of LogSIE_k

$$f(t) = \int_{-1}^1 L(x, t) f(x) dx, \quad x, t \in [-1, 1], \quad (3.84)$$

where, $L(x,t)$ is discribed in Eq. (3.65) and $f(t)$ is an unknown function.

The solution of a homogeneous case is a trivial solution i.e $f(t) = 0$

3.4.4 Solution of Special Singular Integral Equation of Order k

In this section, we will begin our study of Special Singular Integral Equation of order k (SSIEk), which has in form:

$$\int_{-1}^1 \frac{f(x)}{t^k - x^k} dx = g(t^k), \quad t \in (-1, 1) \quad (3.85)$$

where k is a positive odd integer. In particular, when $k = 1$, Eq.(3.85) reduces to the standard airfoil equation in aerodynamics and it has four types of analytical solutions (for more details see Lifanov (1996)).

Let us consider the linear SSIEk of the first kind:

$$\int_{-1}^1 L(x,t)f(x)dx = g(t^k), \quad x \in [-1, 1] \quad (3.86)$$

where the right hand side $g(t^k)$ is known, and $L(x,t)$ is given by

$$L(x,t) = \frac{1}{x^k - t^k}, \quad k \text{ is positive odd integers} \quad (3.87)$$

and need to be determined $f(x)$. The method for solving Eq.(3.86) is based on expansions of the approximate solution in certain EGFs of the first and second kinds $Z_{(i,n)}^k(x)$, $i = \{1, 2\}$ with two cases:

Case (I):

The solution is unbounded at both end-points $x = \pm 1$. For the unbounded solution of Eq.(3.86), we impose the following condition:

$$\int_{-1}^1 f(x)dx \in c, \quad c \text{ is constant.} \quad (3.88)$$

We approximate $f(t)$ by

$$f_n(t) = a_0 w_{(1,k)}(t) Z_{(1,0)}^k(t) + w_{(1,k)}(t) \sum_{j=1}^n a_j Z_{(1,j)}^k(t), \quad (3.89)$$

where $j = 0, 1, \dots, n$; k is positive odd integers.

Substitute Eq. (3.89) into Eq. (3.86) we have:

$$\begin{aligned}
& \int_{-1}^1 L(x, t) f(x) dx = g(t^k) \\
& \int_{-1}^1 \frac{1}{x^k - t^k} w_{(1,k)}(x) \sum_{j=0}^n a_j Z_{(1,j)}^k(x) dx = g(t^k) \\
& \sum_{j=0}^n \int_{-1}^1 \frac{w_{(1,k)}(x)}{x^k - t^k} a_j Z_{(1,j)}^k(x) dx = g(t^k)
\end{aligned} \tag{3.90}$$

$$\begin{aligned}
& a_0 \int_{-1}^1 \frac{w_{(1,k)}(x)}{x^k - t^k} Z_{(1,0)}^k(x) dx + a_1 \int_{-1}^1 \frac{w_{(1,k)}(x)}{x^k - t^k} Z_{(1,1)}^k(x) dx + a_2 \int_{-1}^1 \frac{w_{(1,k)}(x)}{x^k - t^k} Z_{(1,2)}^k(x) dx + \dots \\
& \quad + a_n \int_{-1}^1 \frac{w_{(1,k)}(x)}{x^k - t^k} Z_{(1,n)}^k(x) dx = g(t^k) \\
& a_1 \int_{-1}^1 \frac{w_{(1,k)}(x)}{x^k - t^k} Z_{(1,1)}^k(x) dx + a_2 \int_{-1}^1 \frac{w_{(1,k)}(x)}{x^k - t^k} Z_{(1,2)}^k(x) dx + a_3 \int_{-1}^1 \frac{w_{(1,k)}(x)}{x^k - t^k} Z_{(1,3)}^k(x) dx + \dots \\
& \quad + a_n \int_{-1}^1 \frac{w_{(1,k)}(x)}{x^k - t^k} Z_{(1,n)}^k(x) dx = g(t^k)
\end{aligned} \tag{3.91}$$

k is positive odd integers. It should be noted that:

$$\int_{-1}^1 \frac{w_{(1,k)}(x)}{x^k - t^k} Z_{(1,0)}^k(x) dx = 0$$

Here the first coefficient a_0 is arbitrary. To get n variables with n unknown coefficients, we assume that the $a_0 = 0$. Use Lemma 3.3.3 (vii) to get:

$$\frac{\pi}{k} a_1 Z_{(2,0)}^k(t) + \frac{\pi}{k} a_2 Z_{(2,1)}^k(t) + \frac{\pi}{k} a_3 Z_{(2,2)}^k(t) + \dots + \frac{\pi}{k} a_{n-1} Z_{(2,n-2)}^k(t) + \frac{\pi}{k} a_n Z_{(2,n-1)}^k(t) = g(t^k) \tag{3.92}$$

To determine the unknown coefficients $a_j, j = 1, 2, 3, \dots, n$, we use the classical Galerkin method. To get n unknown and the same number of equations we multiply both sides of Eq. (3.92) by $Z_{(2,j)}^k(t), j = 0, 1, 2, \dots, n-1$, under the weight function $w_{(2,k)}(t)$ then integrating over the interval $[-1, 1]$, Eq. (3.92) becomes:

$$\begin{aligned}
& \frac{\pi}{k} a_1 \int_{-1}^1 w_{(2,k)}(x) Z_{(2,0)}^k(t) Z_{(2,j)}^k(t) dt + \frac{\pi}{k} a_2 \int_{-1}^1 w_{(2,k)}(x) Z_{(2,1)}^k(t) Z_{(2,j)}^k(t) dt + \dots \\
& \quad + \frac{\pi}{k} a_n \int_{-1}^1 w_{(2,k)}(x) Z_{(2,n-1)}^k(t) Z_{(2,j)}^k(t) dt \\
& \quad = \int_{-1}^1 w_{(2,k)}(x) g(t^k) Z_{(2,j)}^k(t) dt
\end{aligned} \tag{3.93}$$

for $j = 0, 1, 2, 3, \dots, n-1$.

For $j = 0$

$$\begin{aligned} \frac{\pi}{k} a_1 \int_{-1}^1 w_{(2,k)}(x) Z_{(2,0)}^k(t) Z_{(2,0)}^k(t) dt + \frac{\pi}{k} a_2 \int_{-1}^1 w_{(2,k)}(x) Z_{(2,1)}^k(t) Z_{(2,0)}^k(t) dt + \cdots \\ + \frac{\pi}{k} a_n \int_{-1}^1 w_{(2,k)}(x) Z_{(2,n-1)}^k(t) Z_{(2,0)}^k(t) dt \\ = \int_{-1}^1 w_{(2,k)}(x) g(t^k) Z_{(2,0)}^k(t) dt \end{aligned} \quad (3.94)$$

Applying the orthogonality property of ECPs of the second kind $Z_{(2,j-1)}^k(t)$, Lemma 3.3.3, Eq. (3.94) yields

$$\begin{aligned} \frac{\pi}{k} \frac{\pi}{2k} a_1 + 0 + \cdots + 0 &= \int_{-1}^1 w_{(2,k)}(x) g(t^k) Z_{(2,0)}^k(t) dt \\ \frac{\pi^2}{2k^2} a_1 &= \int_{-1}^1 w_{(2,k)}(x) g(t^k) Z_{(2,0)}^k(t) dt \end{aligned}$$

For $j = 1$

$$\begin{aligned} \frac{\pi}{k} a_2 \int_{-1}^1 w_{(2,k)}(x) Z_{(2,0)}^k(t) Z_{(2,1)}^k(t) dt + \frac{\pi}{k} a_2 \int_{-1}^1 w_{(2,k)}(x) Z_{(2,1)}^k(t) Z_{(2,1)}^k(t) dt + \cdots \\ + \frac{\pi}{k} a_n \int_{-1}^1 w_{(2,k)}(x) Z_{(2,n-1)}^k(t) Z_{(2,1)}^k(t) dt \\ = \int_{-1}^1 w_{(2,k)}(x) g(t^k) Z_{(2,1)}^k(t) dt \end{aligned} \quad (3.95)$$

Using orthogonality property Lemma 3.3.3 (vi), Eq. (3.95) becomes:

$$\begin{aligned} \frac{\pi}{k} \frac{\pi}{2k} a_2 + 0 + \cdots + 0 &= \int_{-1}^1 w_{(2,k)}(x) g(t^k) Z_{(2,1)}^k(t) dt \\ \frac{\pi^2}{2k^2} a_2 &= \int_{-1}^1 w_{(2,k)}(x) g(t^k) Z_{(2,1)}^k(t) dt \end{aligned}$$

Proceeding in the same way, we obtain:

For $j = n - 1$

$$\begin{aligned} \frac{\pi}{k} a_1 \int_{-1}^1 w_{(2,k)}(x) Z_{(2,0)}^k(t) Z_{(2,n-1)}^k(t) dt + \frac{\pi}{k} a_2 \int_{-1}^1 w_{(2,k)}(x) Z_{(2,1)}^k(t) Z_{(2,n-1)}^k(t) dt + \cdots \\ + \frac{\pi}{k} a_n \int_{-1}^1 w_{(2,k)}(x) Z_{(2,n-1)}^k(t) Z_{(2,n-1)}^k(t) dt \\ = \int_{-1}^1 w_{(2,k)}(x) g(t^k) Z_{(2,n-1)}^k(t) dt \end{aligned} \quad (3.96)$$

Applying orthogonality property Lemma 3.3.3 (vi), Eq. (3.96) becomes:

$$\begin{aligned} 0 + 0 + \cdots + \frac{\pi}{k} \frac{\pi}{2k} a_n &= \int_{-1}^1 w_{(2,k)}(x) g(t^k) Z_{(2,n-1)}^k(t) dt \\ \frac{\pi^2}{2k^2} a_n &= \int_{-1}^1 w_{(2,k)}(x) g(t^k) Z_{(2,n-1)}^k(t) dt \end{aligned}$$

The linear simultaneous equations generated from Eq. (3.94) - (3.96) formulate a system of linear algebraic equations which can be represented as a matrix form:

$$\begin{bmatrix} \frac{\pi^2}{2k^2} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{\pi^2}{2k^2} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \frac{\pi^2}{2k^2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{\pi^2}{2k^2} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_i \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} C_1^k \\ C_2^k \\ \vdots \\ C_i^k \\ \vdots \\ C_n^k \end{bmatrix} \quad (3.97)$$

where $j = 1, 2, \dots, n$, and

$$C_j^k = \langle g, Z_{(2,j-1)}^k \rangle_{w_{(2,k)}(x)} \quad (3.98)$$

Undetermining coefficients $a_j, j = 1, 2, 3, \dots, n-1$ can be found by the relations

$$a_j = \frac{2k^2}{\pi^2} \langle g, Z_{(2,j-1)}^k \rangle_{w_{(2,k)}(x)} \quad (3.99)$$

Case (II): The solution is bounded at both end-points $x = \pm 1$.

For the bounded solution of Eq.(3.86), we impose the following condition:

$$\int_{-1}^1 w_{(1,k)}(x) g(x^k) dx = 0, \quad (3.100)$$

Proceeding in the same way approximates $f(t)$ by:

$$f_n(t) = a_0 w_{(2,k)}(t) Z_{(2,0)}^k + w_{(2,k)}(t) \sum_{j=1}^n a_j Z_{(2,j)}^k(t) \quad (3.101)$$

where $j = 1, \dots, n, k$ is positive odd integer.

Substitute Eq. (3.101) into Eq. (3.86) we have:

$$\begin{aligned}
& \int_{-1}^1 L(x, t) f(x) dx = g(t^k) \\
& \int_{-1}^1 \frac{1}{x^k - t^k} w_{(2,k)}(x) a_0 Z_{(2,0)}^k(x) dx + \int_{-1}^1 \frac{1}{x^k - t^k} w_{(2,k)}(x) \sum_{j=1}^n a_j Z_{(2,j)}^k(x) dx = g(t^k) \\
& a_0 \int_{-1}^1 \frac{1}{x^k - t^k} w_{(2,k)}(x) Z_{(2,0)}^k(x) dx + \sum_{j=1}^n a_j \int_{-1}^1 \frac{w_{(1,k)}(x)}{x^k - t^k} Z_{(2,j)}^k(x) dx = g(t^k)
\end{aligned} \tag{3.102}$$

$$\begin{aligned}
& a_0 \int_{-1}^1 \frac{w_{(2,k)}(x)}{x^k - t^k} Z_{(2,0)}^k(x) dx + a_1 \int_{-1}^1 \frac{w_{(2,k)}(x)}{x^k - t^k} Z_{(2,1)}^k(x) dx + a_2 \int_{-1}^1 \frac{w_{(2,k)}(x)}{x^k - t^k} Z_{(2,2)}^k(x) dx + \dots \\
& + a_n \int_{-1}^1 \frac{w_{(2,k)}(x)}{x^k - t^k} Z_{(2,n)}^k(x) dx = g(t^k)
\end{aligned} \tag{3.103}$$

k is positive odd integers. It should be noted that

$$\int_{-1}^1 \frac{w_{(1,k)}(x)}{x^k - t^k} Z_{(1,0)}^k(x) dx \neq 0.$$

Here the Case(II) is different from Case(I) and the coefficient a_0 not arbitrary; in this case, we have $(n+1)$ unknown coefficients.

Use Lemma 3.3.3 (vii) to get

$$-\frac{\pi}{k} a_0 Z_{(1,1)}^k(t) - \frac{\pi}{k} a_1 Z_{(1,2)}^k(t) - \frac{\pi}{k} a_2 Z_{(1,3)}^k(t) - \dots - \frac{\pi}{k} a_{n-1} Z_{(1,n)}^k(t) - \frac{\pi}{k} a_n Z_{(1,n+1)}^k(t) = g(t^k) \tag{3.104}$$

To determine the unknown coefficients $a_j, j = 0, 1, 2, 3, \dots, n$. We use the classical Galerkin method. To get $(n+1)$ unknowns and the same number of equations, we multiply both sides of Eq. (3.104) by $Z_{(1,m)}^k(t), m = 1, 2, \dots, n+1$, under the weight function $w_{(1,k)}(t)$ then integrating over the interval $[-1, 1]$, Eq. (3.104) becomes:

$$\begin{aligned}
& -\frac{\pi}{k} a_0 \int_{-1}^1 w_{(1,k)}(x) Z_{(1,1)}^k(t) Z_{(1,m)}^k(t) dt - \frac{\pi}{k} a_1 \int_{-1}^1 w_{(1,k)}(x) Z_{(1,2)}^k(t) Z_{(1,m)}^k(t) dt - \dots \\
& - \frac{\pi}{k} a_n \int_{-1}^1 w_{(1,k)}(x) Z_{(1,n+1)}^k(t) Z_{(1,m)}^k(t) dt \\
& = \int_{-1}^1 w_{(1,k)}(x) g(t^k) Z_{(1,m)}^k(t) dt
\end{aligned} \tag{3.105}$$

for $m = 1, 2, 3, \dots, n+1$.

For $m = 1$

$$\begin{aligned}
 & -\frac{\pi}{k}a_0 \int_{-1}^1 w_{(1,k)}(x)Z_{(1,1)}^k(t)Z_{(1,1)}^k(t)dt - \frac{\pi}{k}a_1 \int_{-1}^1 w_{(1,k)}(x)Z_{(1,2)}^k(t)Z_{(1,1)}^k(t)dt - \dots \\
 & \quad - \frac{\pi}{k}a_n \int_{-1}^1 w_{(1,k)}(x)Z_{(1,n+1)}^k(t)Z_{(1,1)}^k(t)dt \\
 & \quad = \int_{-1}^1 w_{(1,k)}(x)g(t^k)Z_{(1,1)}^k(t)dt
 \end{aligned} \tag{3.106}$$

It should be noted that we begin with $m = 1$ since Eq. (3.106) not including the term

$$\int_{-1}^1 w_{(1,k)}(x)Z_{(1,0)}^k(t)Z_{(1,0)}^k(t)dt$$

This fact comes from the integral operator in Lemma 3.3.3 (vii) make increasing order one, which means integral operator in Lemma 3.3.3 vii acts on ECPs of the second kind with (n) give ECPs of the first kind with $(n+1)$. Applying Orthogonality property of Extended Chebyshev polynomials of the first kind $Z_{(1,m)}^k(t)$, $m = 1, 2, 3, \dots, n+1$. Lemma 3.3.3, Eq. (3.106) yields

$$\begin{aligned}
 & -\frac{\pi}{k} \frac{\pi}{2k} a_0 - 0 - \dots - 0 = \int_{-1}^1 w_{(1,k)}(x)g(t^k)Z_{(1,0)}^k(t)dt \\
 & \frac{\pi^2}{2k^2} a_0 = - \int_{-1}^1 w_{(1,k)}(x)g(t^k)Z_{(1,1)}^k(t)dt
 \end{aligned}$$

For $m = 2$

$$\begin{aligned}
 & -\frac{\pi}{k}a_0 \int_{-1}^1 w_{(1,k)}(x)Z_{(1,1)}^k(t)Z_{(1,2)}^k(t)dt - \frac{\pi}{k}a_1 \int_{-1}^1 w_{(1,k)}(x)Z_{(1,2)}^k(t)Z_{(1,2)}^k(t)dt - \dots \\
 & \quad - \frac{\pi}{k}a_n \int_{-1}^1 w_{(1,k)}(x)Z_{(1,n+1)}^k(t)Z_{(1,2)}^k(t)dt \\
 & \quad = \int_{-1}^1 w_{(1,k)}(x)g(t^k)Z_{(1,2)}^k(t)dt
 \end{aligned} \tag{3.107}$$

Using the orthogonality property Lemma 3.3.3 (v) , Eq. (3.107) becomes

$$-0 - \frac{\pi}{k} \frac{\pi}{2k} a_1 - 0 - \dots - 0 = \int_{-1}^1 w_{(1,k)}(x) g(t^k) Z_{(1,0)}^k(t) dt$$

$$\frac{\pi^2}{2k^2} a_1 = - \int_{-1}^1 w_{(1,k)}(x) g(t^k) Z_{(1,1)}^k(t) dt$$

Proceeding in the same way, we obtain:

For $m = n + 1$

$$-\frac{\pi}{k} a_0 \int_{-1}^1 w_{(1,k)}(x) Z_{(1,1)}^k(t) Z_{(1,n+1)}^k(t) dt - \frac{\pi}{k} a_1 \int_{-1}^1 w_{(1,k)}(x) Z_{(1,2)}^k(t) Z_{(1,n+1)}^k(t) dt - \dots$$

$$-\frac{\pi}{k} a_n \int_{-1}^1 w_{(1,k)}(x) Z_{(1,n+1)}^k(t) Z_{(1,n+1)}^k(t) dt$$

$$= \int_{-1}^1 w_{(1,k)}(x) g(t^k) Z_{(1,n+1)}^k(t) dt \quad (3.108)$$

Applying the orthogonality property Lemma 3.3.3 (v) , Eq. (3.108) becomes:

$$-0 - 0 - \dots - \frac{\pi}{k} \frac{\pi}{2k} a_n = \int_{-1}^1 w_{(2,k)}(x) g(t^k) Z_{(2,1)}^k(t) dt$$

$$\frac{\pi^2}{2k^2} a_n = - \int_{-1}^1 w_{(1,k)}(x) g(t^k) Z_{(2,n+1)}^k(t) dt$$

The linear simultaneous equations generated from Eq. (3.106) - (3.108) formulate a system of linear algebraic equations with $(n + 1)$ unknown coefficients:

$$a_m = - \frac{k^2}{\pi^2} \langle g, Z_{(1,m+1)}^k \rangle_{w_{(1,k)}}, \quad m = 0, 1, 2, 3, \dots, n \quad (3.109)$$

We use Matlab to determine coefficients $a_m, m = 0, 1, 2, 3, \dots, n$.

3.5 Conclusion

In this chapter, the definition of the Extended Chebyshev polynomials of the first and second kinds of order k , $Z_{(i,n)}^k(x), i = \{1, 2\}$ will be presented. It showed some graphs of the ECPs of the first and second kinds of order k . The zeros of ECPs of the first and second kinds of order k are discussed and described. The behavior of Extended Chebyshev

sequences are explained when k is sufficient large odd integers. It is found that Extended Chebyshev sequences of the first and second kinds converge approximately into one of the three values $\{-1, 0, 1\}$ depending on index n .

We found that ECPs of the first and second kinds of order k , $Z_{(i,n)}^k(x)$, $i = \{1, 2\}$ are orthogonal with respect to weight functions $w_{(1,k)}(x) = \frac{x^{k-1}}{\sqrt{1-x^{2k}}}$, $w_{(2,k)}(x) = x^{k-1}\sqrt{1-x^{2k}}$ respectively on the interval $[-1, 1]$, where k is positive odd integer.

We stated some of the standard formulae of the two kinds recurrence relations and the relationships between each other, then we described different kinds of ECPs of order k and some of its associated relations.

Spectral properties ECPs of the first and second kinds $Z_{(i,n)}^k(x)$, $i = \{1, 2\}$ are proved. We also demonstrated how to use these orthogonal polynomials to aid in the solution of linear homogenous and inhomogeneous logarithmic singular integral equations (LogSIEk). The density functions are characterized in terms of truncated series of ECPs of the first and second kinds. Numerical procedures are derived to evaluate the density functions. Moreover, a study of effective and accurate approximate methods for solving the SSIEk of the first kind into two cases over a finite interval $[-1, 1]$ are presented.