CHAPTER 1

INTRODUCTION

1.1 Motivation and Overview

Many phenomena of almost all practical applications, such as physical systems, science economics, engineering, electrical network analysis and medicine are represented as mathematical models, some of these models can be formulated as integral equations (IEs). Since, in many cases, IEs cannot be solved analytically, it is required to obtain approximate solutions. There are many different methods to find approximate solutions of IEs, for examples can be found in Polyanin & Manzhirov (2008), Porter & Stirling (1990) and Atkinson (1997).

Recently, a major concern in modern engineering is the development of mathematical models that appropriately represent the problem under study. For instance, the complicated real physical situation with boundary conditions under load are given. These types of phenomena can be reduced into IEs. Finding the analytical solution of the IEs, we are facing great difficulties and limitations. To overcome this problem, mathematical models are utilized to represent the problem properly. For the approximate solution, numerical methods have to be used, for examples see Corduneanu (1991) and Hochstadt (2011).

One of the famous approximate methods is to use orthogonal function to represent the unknown functions which is needed to be deferminated. The main characteristic of this technique is to reduce IEs into the systems of algebraic equations, that is easy to solve (Araghi & Noeiaghdam, 2016a). Orthogonal function methods are developed for the numerical solution of singular integral equations (SIEs) with Cauchy kernel, by considering an appropriate weight function (Hsiao & Chen, 1979). Special attention has been given to the application of block-pulse functions (Sahlan et al., 2015), Legendre polynomials (Jafarian & Esmailzadeh, 2013), Laguerre-Gaussians quadrature formula (Aboiyar et al., 2010), Chebyshev polynomials (Paryab & Rostami, 2008).

The most important special types of IEs is the SIEs. The numerical solution of the SIEs can be computed using multiple methods, such as collocation and Galerkin methods, for more details see Lifanov (1996) and Golberg (1979). Indeed, both collocation and Galerkin methods begin in the same way, the two methods differ in the way in which the approximation solution is chosen from the suitable finite dimensional linear space. In the collocation method, IEs are considered in a finite dimensional space, so it is evaluated at some discrete points in which it yields to a linear system of equations. For Galerkin method, IEs are checked by appropriate base functions which lead to a linear system of equations (Golberg, 2013). Many researchers have proposed different methods to solve SIEs and hyper-singular integral equations (HSIEs), some references can be found in Ervin & Stephan (1992), Iovane & Ciarletta (2003), Chakrabarti & Manam (2003), Mahiub et al. (2011) and Dardery & Allan (2014).

The classic formulation of the SIEs have been used in several areas, such as mechanics, plates, shells, soil mechanics, potential problems, Bukhari (2009) and Kress & Lee (2003). SIEs are connected with such boundary value problems in a natural way, Lage & Schwab (1999) and Chakrabarti & George (1994).

However, the theory of SIEs with their integral in the sense of its principal value was aroused directly after the development of the classical theory of IEs by Fredholm in 1903. SIEs and their properties were investigated by many authors like Lifanov (1996), Kumar & Sangal (2004), Jen & Srivastav (1981), Kythe & Schäferkotter (2004), Tuck (1980), Atkinson (1997) and Muskhelishvili & Radok (2008).

The Cauchy Singular Integral Equations (CSIEs) as a mathematical tool occur in several fields of science such as applied mathematics and physics, mechanics, economics, and medicine. Numerous applications of SIEs in theory of elasticity, plasticity, aerodynamics, filtration theory, hydromechanics engineering, fluid dynamics and electrical engineering have been extensively studied by several authors like Atkinson (1997), Bialecki & Stenger (1988) and Mennouni (2013). Orthogonal Polynomials play an important role in numerical analysis and it can be used to model of science and describe real-world phenomena, this family of polynomials are rich and useful. However, Special singulaur IEs of Log and cauchy types are not much elaborated. Generlization of orthogonal polynomials and its properties lead to find approximate solutions of the above problemes.

Furthermore, many boundary value problems which arise in analysis and in many problems in mathematical physics, mechanics and engineering, such as potential and scattering theories can be reduced to the Logarithmic Singular Integral Equations (LSIEs).

Particularly, problems in the elasticity theory, aerodynamics, thermoplasticity and classical hydraulics problems especially those involved to the determination of a free surface are under non-linear boundary conditions Gakhov (1963), Boyd (2001).

On the other hand, the advancement and innovation of technology and numerical modeling of engineering problems have been experiencing constant technological developments, as well as an enlargement of their scope for its efficiency and versatility in order to obtain accurate and efficient results.

1.2 Objectives

This thesis specifically aims:

(i) To develop new classes of one dimensional orthogonal polynomials.

(ii) To solve special class logarithmic singular integral equations (first and second kinds) of order k (LogSIEk) using a new class of orthogonal polynomials.

(iii) To solve special class singular integral equations of the first kind of order k (SSIEk) using a new class of orthogonal polynomials.

(iv) To obtain numerical results for LogSIEk, SSIEk and compare the approximate solution in the proposed method with the exact solution and other methods.

1.3 Contributions

The main contribution of this thesis is the development of orthogonal polynomials of four kinds named Extended Chebyshev polynomials (ECPs) polynomials of the first, second, third and fourth kinds of order k, denote it as $Z^{k}_{(i,n)}(x), i = \{1,2,3,4\}$, where k is positive odd integer, $n = 0,1,2,\ldots$ corresponding weight functions of order k in one dimension $(w_{(i,k)}(x), i = \{1,2,3,4\})$, and their use to solve SIEs. The main characteristics of the behavior Extended Chebyshev sequences when k is sufficient large odd integers $\{Z^{2m+1}_{(i,n)}(x)\}^{k}_{m=0}$; $i = \{1,2,3,4\}; n = 0,1,2,3,\ldots$ are discussed and proved. Moreover, the zeros of all kinds of Extended Chebyshev $Z^{k}_{(i,n)}(x), i = \{1,2,3,4\}$ are described and discussed.

Essentially, this thesis focuses on two types of SIEs in one dimension:

(i) Special logarithmic singular integral equations of order k (LogSIEk),

(ii) Special kind of singular integral equations (SSIEk), and their associated IEs.

New expressions are found and a better understanding of the involved techniques is achieved, it is also of interest to derive expressions for the solution of the eigenvalue problem, eigenvalues and corresponding eigenfunctions are found for special logarithmic singular integral equations of the first kind of order k (LogSIEk).

The unknown functions are characterized in terms of truncated series of Extended Chebyshev Polynomials, and their properties. Efficient numerical procedures are derived to evaluate the unknown functions everywhere and on all the values of interest; the goal is to give a numerically effective and efficient expression for the unknown function. Moreover, a study of effective and accurate approximate methods for solving the SSIEk of the first kind over a finite interval $[-1,1]$ are presented.

Another important contribution is the proper understanding of the orthogonality properties for four kinds with weight functions of order k. In the solution, ECPs of the first kind $Z^k_{(1,n)}(x)$, second kind $Z^k_{(2,n)}(x)$, third kind $Z^k_{(3,n)}(x)$, and fourth kind $Z^k_{(4,n)}(x)$, corresponding to respective weight functions $w_{(1,k)}(x) = \frac{x^{k-1}}{\sqrt{1-x^{2k}}}$; $w_{(2,k)}(x) = x^{k-1}\sqrt{1-x^{2k}}$; $w_{(3,k)}(x) = x^{k-1}\sqrt{\frac{1+x^k}{1-x^k}}$; $w_{(4,k)}(x) = x^{k-1}\sqrt{\frac{1-x^k}{1+x^k}}$ on the interval $[-1,1]$ have been used to obtain systems of linear algebraic equations; these systems are solved numerically. The developed techniques are to be programmed in MATLAB, implementing benchmark problems to test these calculations and the computations.

Another contribution is the development of computational program to solve the considered problems, and the numerical results that are obtained. The programming is in general not easy and requires a careful treatment of the involved singular integrals to build the full matrices that arise from the base polynomials.

1.4 Thesis Organization

To fulfill the objectives, this thesis is structured in six chapters and two appendices. Each chapter and each appendix is divided into sections and further into subsections in order to display the contents in the hopefully most clear and accessible way. Each one starts with a short introduction that sheds light on its contents. A list of references is also included at the end of thesis.

Chapter 1: The current chapter presents a general introduction to the thesis. The more general aspects are described and discussed; the framework that connects its different parts is described. It includes the introduction, the objectives, the contribution and the thesis outline.

Chapter 2: The aim of this chapter is to give a detailed overview of the studies which show how the SIEs are solved, having main reviewed the articles of some researchers. Literature review is presented, discussing the Chebyshev polynomials of the first - fourth kinds. Main properties and theorems are considered, the singular IEs with Cauchy kernels in one dimensions, using IE techniques. The detailed history of the IEs and its applications are also included along with emphasis and the identification of some topics related to IEs.

Chapter 3 includes the first main contribution of this thesis; particularly, we defined the Extended Chebyshev polynomials of the first and second kinds of order k, $Z^{k}_{(i,n)}(x), i = \{1,2\}$. It is shown some of respectives graphs of the ECPs of the first and second kinds of order k. The zeros of ECPs of the frist and second kinds of order k are discussed and described. The behavior of Extended Chebyshev sequences are explained when k is sufficient large odd integers. It is found that ECPs sequences of first and second kinds converge approximately into one of the three values $\{-1,0,1\}$ depending on index n.

We found that ECPs of the first and second kinds of order k, $Z^{k}_{(i,n)}(x) i = \{1,2\}$ are orthogonal with respect to weight functions $w_{(1,k)}(x) = \frac{x^{k-1}}{\sqrt{1-x^{2k}}}$, $w_{(2,k)}(x) = x^{k-1}\sqrt{1-x^{2k}}$ respectively on the interval $[-1,1]$, where k is positive odd integer.

We state some of the standard formulae of the two kinds recurrence relations and the relationships between the other two kinds are proved. Then we describe different kinds of ECPs with order k and some of its associated relations.

Spectral properties ECPs of the first and second kind $Z^{k}_{(i,n)}(x), i = \{1,2\}$ are proved, they will be used in greater detail in this chapter of the thesis. Moving into the applications ECPs of the first and second kind with order k, we also demonstrate how to use these orthogonal polynomials to aid in the solution of linear homogenous and inhomogeneous logarithmic singular integral equations (LogSIEk). The density functions are characterized in terms of truncated series of ECPs of the first and second kind. Effective numerical procedures are derived to evaluate the density functions everywhere and on all the values of interest. Moreover, a study of effective and accurate approximate methods for solving the SSIEk of the first kind into two cases over a finite interval $[-1,1]$ are presented.

Chapter 4 contains the second main contribution of this thesis, especially, we defined ECPs of the third and fourth kinds of order k, $Z^k_{(i,n)}(x), i=\{3,4\}$. It show some of graphs of the ECPs of the of third and fourth kinds of order k. The zeros of ECPs of the third and fourth kinds of order k are discussed and described.

Recurrence relations and the relations between the other two kinds are proved. We state some of the associated relations of the two kinds. The behavior of Extended Chebyshev sequences of third and fourth kinds are described when k is sufficient large odd integers, It is found that Extended Chebyshev sequences of third and fourth kinds converge approximately into two values $\{-1,1\}$ depending on index n.

It is found that ECPs of the third and fourth kinds $Z^k_{(i,n)}(x), i=\{3,4\}$ are orthogonal with respect to weight functions, $w_{(3,k)}(x) = x^{k-1}\sqrt{\frac{1+x^k}{1-x^k}}$ and $w_{(4,k)}(x) = x^{k-1}\sqrt{\frac{1-x^k}{1+x^k}}$ respectively, on the interval [-1,1], where k is positive odd integers. Spectral properties ECPs of the third and fourth kinds $Z^k_{(i,n)}(x), i=\{3,4\}$ are proved. They will be used in greater detail in this chapter of the thesis. Moving into the applications ECPs of the third and fourth kinds of order k, we also demonstrate how to use these orthogonal polynomials to aid in the solution of SSIEk. The unknown functions are characterized in terms of truncated series of ECPs. Effective numerical methods are proposed to evaluate the density functions. Furthermore, a study of effective and accurate approximate methods for solving the SSIEk of the first kind with third and fourth cases over a finite interval [-1, 1] is presented.

Chapter 5 shows computational results of standard examples and compares the results with those obtained previously. In addition to standard examples which are solved for purpose of comparison, the computational results are presented. Diagrams based on the exact and approximate solutions are presented for LogSIEk and SSIEk with four cases.

Chapter 6 incorporates the conclusion of the thesis, including a short discussion on the results and some perspectives for future work. It is followed by the appendices and the list of bibliographies.