CHAPTER 6

CONCLUSIONS AND FUTURE WORK

This chapter is divided into two sections: First, we present the conclusion of this work. Second, we sketch the outlines of future work:

6.1 Conclusions

The first part of this thesis has proposed a new class of orthogonal polynomials named Extended Chebyshev polynomials of the first, second, third and fourth kinds of order k $Z^k_{(i,n)}(x), i = \{1,2,3,4\}$, k is positive odd integrs, which is denoted by:

(i) Extended Chebyshev polynomials of the first kind of order k, $Z^k_{(1,n)}(x)$,

(ii) Extended Chebyshev polynomials of the second kind of order k, $Z^k_{(2,n)}(x)$,

(iii) Extended Chebyshev polynomials of the third kind of order k, $Z^k_{(3,n)}(x)$,

(iv) Extended Chebyshev polynomials of the fourth kind of order k, $Z^k_{(4,n)}(x)$,

Moreover, the Extended Chebyshev polynomials of the first, second, third and fourth kinds of order k have precisely n zeros which are related with value of k so that

$$x_{(i,m)} = \begin{cases} \sqrt[k]{\cos \frac{(m-\frac{1}{2})\pi}{n}}, & i=1 \\ \sqrt[k]{\cos \frac{m\pi}{n+1}}, & i=2 \\ \sqrt[k]{\cos \frac{m\pi}{n+1}}, & i=3 \\ \sqrt[k]{\cos \frac{m\pi}{n+\frac{1}{2}}}, & i=4 \end{cases}$$

where $x_{(i,m)}, i=\{1,2,3,4\}; m=1,2,3,\ldots,n$ denoted to the zeros of Extended Chebyshev polynomials of the first, second, third and fourth kinds of order k respectively.

We described the main characteristics of the Extended Chebyshev sequences when k is sufficient large odd integers $\{Z_{(i,n)}^{2m+1}(x)\}_{m=0}^{k}$; $i=\{1,2,3,4\}; n=0,1,2,3,\ldots$. The behavior of Extended Chebyshev sequences are discussed when k is sufficient large odd integers. It was found that Extended Chebyshev sequences of the first and second kinds converge approximately to one of the three values $\{-1,0,1\}$, depending on index n, Eq. (6.1):

$$Z_{(i,n)}^{k}(x) \approx \begin{cases} 0 & i=1,2; \quad n=odd \\ -1 & i=1,2; \quad n=2(2m+1) \\ 1 & i=1,2; \quad n=4(m) \end{cases} \tag{6.1}$$

where $m=0,1,2,3,\ldots$.

For Extended Chebyshev sequences of the third and fourth kinds, it was found approximately converging to one of the two values $\{-1,1\}$ Eq. (6.2) - (6.3):

$$Z_{(3,n)}^{k}(x) \approx \begin{cases} -1 & n=1,2,5,6,9,10\ldots \\ 1 & n=0,3,4,7,8,11,12\ldots \end{cases} \tag{6.2}$$

$$Z_{(4,n)}^{k}(x) \approx \begin{cases} -1 & n=2,3,6,7,10,11,\ldots \\ 1 & n=0,1,4,5,8,9,12,\ldots \end{cases} \tag{6.3}$$

We found that Extended Chebyshev polynomials of the first - fourth kinds $Z_{(i,n)}^{k}(x), i=\{1,2,3,4\}$ $n=0,1,2,\ldots$ are orthogonal with respect to weight functions $w_{(1,k)}(x)=$

$\frac{x^{k-1}}{\sqrt{1-x^{2k}}}$, $w_{(2,k)}(x) = x^{k-1}\sqrt{1-x^{2k}}$, $w_{(3,k)}(x) = x^{k-1}\sqrt{\frac{1+x^k}{1-x^k}}$ and $w_{(4,k)}(x) = x^{k-1}\sqrt{\frac{1-x^k}{1+x^k}}$ on the interval $[-1,1]$ respectivily. Table 6.1 shows the summaries of the orthogonality properties of Extended Chebyshev polynomials of first - fourth kinds.

Table 6.1: Orthogonality of Extended Chebyshev polynomials

$\langle Z^k_{(i,n)}, Z^k_{(i,m)}\rangle_{w(i,k)}$			
i	$m=n=0$	$m=n$	$m\neq n$
1	$\frac{\pi}{k}$	$\frac{\pi}{2k}$	0
2	$\frac{\pi}{2k}$	$\frac{\pi}{2k}$	0
3	$\frac{\pi}{k}$	$\frac{\pi}{k}$	0
4	$\frac{\pi}{k}$	$\frac{\pi}{k}$	0

Also, the result is that Extended Chebyshev polynomials of the first-fourth kinds of order k satisfy the three terms of recurrence relations; these relations between Extended Chebyshev polynomials of first - fourth kinds $Z^k_{(i,n)}(x), i = \{1,2,3,4\}$ are proved. Associated relations of Extended Chebyshev polynomials of the first - fourth kinds of order k, are described:

$$Z^k_{(i,n)}(x) = 2x^k Z^k_{(i,n-1)}(x) - Z^k_{(i,n-2)}(x)$$

where $i = \{1,2,3,4\}$; $n = 2,3,\cdots$; k is positive odd integers, $Z^k_{(i,0)}(x)$ and $Z^k_{(i,1)}(x)$ are given.

We discussed the connections between the four kinds of Extended Chebyshev polynomials, and how they are related to recursive techniques. The recursive relationship can be used to compute the value of the Extended Chebyshev polynomials of all $n = 0,1,2,3,\ldots; x \in [-1,1]$ by knowing only the first two terms of the Extended Chebyshev polynomials of order k. The main properties of $Z^k_{(i,n)}(x), i = \{1,2,3,4\}$, can be summarized as follows:

(i) The relationship between Extended Chebyshev polynomials of the first and second kinds:

$$Z^k_{(2,n)}(x) - Z^k_{(2,n-2)}(x) = 2Z^k_{(1,n)}(x) \tag{6.4}$$

(ii) The relations between Extended Chebyshev polynomials of the second, third and fourth kinds of order k are:

$$Z^k_{(2,n)}(x) = \frac{1}{2}\left[Z^k_{(3,n)}(x) + Z^k_{(4,n)}(x)\right]$$

(iii) The three terms of recurrence relations between Extended Chebyshev polynomials of the second and third kinds are:

$$Z^k_{(3,n)}(x) = Z^k_{(2,n)}(x) - Z^k_{(2,n-1)}(x)$$

(iv) The three terms of recurrence relations between Extended Chebyshev polynomials of the second and fourth kinds are:

$$Z^k_{(4,n)}(x) = Z^k_{(2,n)}(x) + Z^k_{(2,n-1)}(x)$$

(v) The relations between Extended Chebyshev polynomials of the first, third and fourth kinds are:

$$2Z^k_{(1,n)}(x) = Z^k_{(4,n)}(x) - Z^k_{(4,n-1)}(x) = Z^k_{(3,n)}(x) + Z^k_{(3,n-1)}(x)$$

Main spectral properties of the four kinds $Z^k_{(i,n)}(x), i = \{1,2,3,4\}$ can be briefly shown:

$$\int_{-1}^{1} w_{(i,k)}(x) \frac{Z^k_{(i,n)}(x)}{x^k - t^k} dx = \frac{\pi}{k} \begin{cases} Z^k_{(2,n-1)}(t), & i = 1 \quad n = 1,2,3,\ldots, \\ Z^k_{(1,n+1)}(t), & i = 2 \quad n = 0,1,2,\ldots, \\ Z^k_{(4,n)}(t), & i = 3 \quad n = 0,1,2,\ldots, \\ -Z^k_{(3,n)}(t), & i = 4 \quad n = 0,1,2,\ldots, \end{cases}$$

where $Z^k_{(1,n)}(t), Z^k_{(2,n)}(t), Z^k_{(3,n)}(t)$ and $Z^k_{(4,n)}(t)$ are Extended Chebyshev polynomials of the first, second, third and fourth kinds of order k respectively, and $t \in (-1,1)$.
An important feature of this thesis is the eigenvalue problem, the integral equation

$$\lambda\phi(t) = \int_{-1}^{1} \frac{x^{k-1}}{\sqrt{1-x^{2k}}} \phi(x) \log|x^k - t^k| dx$$

has an eigensolutions $\phi(t) = \phi_n(t) = Z^k_{(1,n)}(t)$ corresponding eigenvalues $\lambda = \lambda_n = -\frac{\pi}{nk}, n = 1,2,3,\ldots$ where $Z^k_{(1,n)}(t)$ is the Extended Chebyshev polynomials of the first kind of order k, k is positive odd integers. Moreover when $n = 0$

$$\int_{-1}^{1} \frac{x^{k-1}}{\sqrt{1-x^{2k}}} Z^k_{(1,0)}(x) \log|x^k - t^k| dx = -\frac{\pi}{k} \log 2$$

Special logarithmic singular integral equations of the first and second kinds of order k (LogSIEk) was found, for nonhomogeneous LogSIEk truncated series kind of Extended Chebyshev polynomials of the first kind are used to find an approximate solution.

Morever, the main emphasis in this thesis has been on the solution of LogSIEk and SSIEk. We used the Extended Chebyshev polynomials $Z^k_{(i,n)}(x), i = \{1,2,3,4\}, n = 0,1,2,3,\ldots$ and there spectral properties to solve:

(i) The special logarithmic singular integral equations of order k (LogSIEk)

(a) Linear LogSIE of the first kind:

$$g(t) = \lambda \int_{-1}^{1} f(x) \log \left| x^k - t^k \right| dx, \ \ k \text{ is positive odd integers}, x,t \in [-1,1]$$

(b) Linear non homogeneous LogSIE of the second kind

$$f(t) = g(t) + \lambda \int_{-1}^{1} f(x) \log |x^k - t^k| dx, \qquad x,t \in [-1,1]$$

where k is positive odd integers.

For homogeneous and nonhomogeneous LogSIEk truncated series Extended Chebyshev polynomials of the first kind are used to find the approximate solution. We investigated the numerical solution for a class of LogSIEk

(ii) Special class of singular integral equations of order k, (SSIEk):

$$\frac{1}{k\pi} \int_{-1}^{1} \frac{f(x)}{x^k - t^k} dx = g(t^k), k \text{ is positive odd integers }, t \in (-1,1) \tag{6.5}$$

The method presented for solving SSIEk is based on expansions of the approximate solution in Extended Chebyshev polynomials $Z^k_{(i,n)}(x), i = \{1,2,3,4\}$ with unknown coefficients with certain conditions into four casas:

- Case (I): The solution is unbounded at both end points $x = \pm 1$,
- Case (II): The solution is bounded at both end points $x = \pm 1$.
- Case (III):The solution is bounded at the end point $x = 1$. but unbounded at $x = 1$.
- Case (IV):The solution is bounded at the end point $x = 1$. but unbounded at $x = -1$.

We found that the unknown function $f(t)$ can be approximate by $f_n(t)$:

$$f_n(t) = w_{(i,k)}(t) Z^k_{(i,0)}(t) + w_{(i,k)}(t) \sum_{j=1}^{n} a_j Z^k_{(i,j)}(t),$$

where $j = 1, \cdots, n; i = \{1,2,3,4\}, k$ is positive odd integer. We obtain a system of linear algebraic equations. The computational performance approach through some numerical examples is investigated. The goal is to give a numerically effective and efficient expression for the unknown function and to derive expressions for the unknown function of the solution of the SSIEk.

MATLAB code performs calculations to obtain numerical results for the numerical examples and compares results with other methods to illustrate the effectiveness and accuracy of the proposed methods. A great deal of effort has been made in the development of MATLAB code for the approximate solution of LogSIEk and SSIEk. Finally, effective computational methods are proposed to solve the linear homogeneous and inhomogeneous LogSIEk.

6.2 Future Work

It is hoped that the work of this thesis will be extended into:

- Computational treatment of discrete Extended Chebyshev polynomials of the first - fourth kinds $Z^k_{(i,n)}(x), i = \{1,2,3,4\}$.
- Extended Chebyshev polynomials can be used for a solution of Cauchy singular fractional integro-differential equation.
- Construct the schemes of differential equations which Extended Chebyshev polynomials are solution to it.
- Approximate solution of linear integro-differential equations in terms of Extended Chebyshev polynomials.
- Extended Chebyshev polynomials can be used as a solution of ordinary differential equations (Approximate solution as a finite series of Extended Chebyshev polynomials)