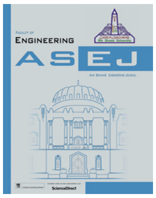
Ain Shams Engineering Journal 11 (2020) 489–494

Contents lists available at ScienceDirect

Ain Shams Engineering Journal

journal homepage: www.sciencedirect.com



A new class of orthogonal polynomials for solving logarithmic singular integral equations

H. Alhawamda [a,*], B.M. Taib [a], Z.K. Eshkuvatov [b], R.I. Ibrahim [a]

[a] *Faculty of Science and Technology (FST), Universiti Sains Islam Malaysia (USIM), Negeri Sembilan, Malaysia*
[b] *Faculty of Ocean Engineering Technology and Informatics, Universiti Malaysia Terengganu (UMT), 21030, Kuala Terengganu, Malaysia*

ARTICLE INFO

Article history:
Received 20 March 2017
Accepted 20 October 2017
Available online 29 November 2019

Keywords:
Chebyshev polynomials
New orthogonal basis functions
Singular integral

ABSTRACT

In this note, we propose a new class of orthogonal polynomials (named Bachok–Hasham polynomials $H^k_{1n}(x)$), which is an extension of the Chebyshev polynomials. Eigenfunctions and corresponding eigenvalues are found for the homogeneous second kind of Logarithmic Singular Integral Equations (LogSIEs). For non-homogeneous LogSIEs truncated series of the first kind Bachok–Hasham polynomials are used to find approximate solution. It is found that first kind of Bachok–Hasham polynomials $(H^k_{1n}(x))$ are orthogonal with weight $w_k(x) = \frac{x^{k-1}}{\sqrt{1-x^{2k}}}$, where k is positive odd integer. Properties of first kind of Bachok–Hasham polynomials are also proved. Finally, two examples are presented to show the validity and accuracy of the proposed method.

1. Introduction

Many phenomena of almost all practical applications, such as physical systems, economics, engineering, electrical network analysis and medicine are represented to the mathematical models, some of these models can be formulated as integral equations. Since, in many cases, integral equations cannot be solved analytically, it is required to obtain approximate solutions. There are many different methods to find approximate solution of integral equations [1–4].

One of the famous approximate method is to use orthogonal function to represent the unknown time functions which is need to be defined. The main characteristic of this technique is to reduce integral equations into the systems of algebraic equations, which is easy to solve. Orthogonal functions method have been proposed to solve linear integral equations of the first and second kind, particularly, applications of Walsh functions [5], block-pulse functions [6], Legendre polynomials [7], Laguerre-Gaussians quadrature formula [8], Chebyshev polynomials [9]. An important type of integral equation that contains a singular kernel, many researchers have proposed different methods to solve singular integral equations (SIEs) and hyper-singular integral equations (HSIEs), approximately [10–18].

Furthermore, the Logarithmic Singular Integral Equations (LogSIEs) arises in analysis and in many problems in mathematical physics, mechanics and engineering such as potential and scattering theories [19–25], particularly, problems in the elasticity theory, aerodynamics, thermoplasticity and classical hydraulics problems especially those involved to the determination of a free surface under non-linear boundary conditions [29–32].

In the present paper, we have defined a new class of orthogonal polynomials of the first kind named Bachok–Hasham polynomials $H^k_{1n}(x)$, where k is odd positive integer with $x \in [-1,1]$ and used it to solve:

- Linear LogSIE of first kind

$$g(t) = \lambda \int_{-1}^{1} f(x) \log|x^k - t^k| dx,$$
$$\text{k is positive odd integer},\ x,t \in [-1,1] \qquad (1)$$

- Linear non homogeneous LogSIE of the second kind

$$f(t) = g(t) + \lambda \int_{-1}^{1} f(x) \log|x^k - t^k| dx,$$
$$\text{k is positive odd integer},\ x,t \in [-1,1] \qquad (2)$$

---

* Corresponding author.
E-mail address: hisham_math77@yahoo.com (H. Alhawamda).

Peer review under responsibility of Ain Shams University.

Production and hosting by Elsevier

https://doi.org/10.1016/j.asej.2017.10.010

We assume that Eqs. (1) and (2) has a unique solution $f(t)$. The method presented here consists of expanding the unknown solution $f(x)$ by first kind of Bachok–Hasham polynomials $H^k_{1n}(x)$ with unknown coefficients and main problem is to find these unknown coefficients.

The structure of the note is arranged as follows: in Section 2, we provide some theoretical aspects of inner product in Hilbert space and prove Theorems and Lemmas. In Section 3, derivation of the proposed method for solving (1) and (2) are presented. Numerical examples are provided in Section 4. Finally, conclusion and acknowledgement are given in Section 5.

2. Preliminaries

2.1. Orthogonal polynomials and weight functions

Definition 1. Two functions $f(x)$ and $g(x)$ in $L_2[a,b]$ are said to be orthogonal on the interval $[a,b]$ with respect to a given continuous and non-negative weight function $w(x)$if

$$\langle f,g\rangle = \int_a^b w(x)f(x)g(x)dx = 0. \tag{3}$$

The formal definition of an inner product of the real variable functions is as follows.

Definition 2. An inner product $\langle .,.\rangle$ is a linear function of elements $f,g,h,$...of a vector space that satisfies the axioms:

- $\langle f,f\rangle = 0$ if and only if $f = 0$,
- $\langle f,g\rangle = \langle g,f\rangle$,
- $\langle f+g,h\rangle = \langle f,g\rangle + \langle f,h\rangle$,
- $\langle \alpha f,g\rangle = \alpha\langle g,f\rangle$for any scalar α.

Inner product defines the norm in L_2space as

$$\|f\| = \|f\|_2 = \sqrt{\langle f,g\rangle}. \tag{4}$$

Theorem 2.1 (*Residue Theorem, Spiegel [26]*). *Let C be closed path within and on which f is holomorphic except for m isolated singularities. Then*

$$\oint_C f(z)dz = 2\pi i\sum_{n=1}^{m} Res(f(z),z_n), \tag{5}$$

and the residue for z_0 pole of order n,

$$Res(f(z),z_n) = \lim_{x\to z_0}\frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\left[(z-z_0)^n f(z)\right]. \tag{6}$$

2.2. The Chebyshev polynomials of first kind

Definition 3. Mason [27] The Chebyshev polynomial $T_n(x)$ of the first kind is a polynomial in x of degree n, defined by the relation

$$T_n(x) = \cos(n\theta), x = \cos(\theta), ; n = 0,1,2,\dots \tag{7}$$

or three term relations

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), n = 1,2,\dots; \quad T_0(x) = 1, T_1(x) = x, \tag{8}$$

It is known that $T_n(x), n = 0,1,\dots$ forms an orthogonal polynomials system on $[-1,1]$ with respect to the weight $w(x) = \frac{1}{\sqrt{1-x^2}}$

$$\langle T_n,T_m\rangle = \int_{-1}^{1}\frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}dx = \begin{cases} 0, & m\neq n,\\ \frac{\pi}{2}, & m=n,\\ \pi, & m=n=0.\end{cases} \tag{9}$$

2.3. Bachok-Hasham polynomials of order k

Definition 4. Bachok-Hasham Polynomials of order k: The Bachok–Hasham polynomials of first kind $H^k_{1n}(x)$ for order k, where k is positive odd integer number, is a polynomial in x of degree (kn), defined by the relation (for fixed k):

$$H^k_{1n}(x) = \cos(n\theta), \quad x^k = \cos(\theta), \quad k = 1,3,5,\dots,N; \quad n = 0,1,2,\dots \tag{10}$$

The range of the variable x is $[-1,1]$ and the range of the corresponding variable θ is $[0,\pi]$. These ranges are traversed in opposite directions i.e. $x = -1$, corresponds to $\theta = \pi$ and $x = 1$ corresponds to $\theta = 0$. The above definition is well defined and for $x\in[-1,1]$ with $|\cos(\theta)|\leqslant 1$, implies $|x|^k\leqslant 1$. Here we illustrate the first few Bachok-Hasham polynomials order k,

$$\begin{aligned}
&H^k_{10}(x) = 1,\\
&H^k_{11}(x) = x^k,\\
&H^k_{12}(x) = 2x^{2k} - 1,\\
&H^k_{13}(x) = 4x^{3k} - 3x^k,\\
&H^k_{14}(x) = 8x^{4k} - 8x^{2k} + 1,\\
&H^k_{15}(x) = x^k(16x^{4k} - 20x^{2k} + 5),\\
&H^k_{16}(x) = 32x^{6k} - 48x^{4k} + 18x^{2k} - 1.
\end{aligned} \tag{11}$$

Its easy to show that when $k = 1$,

$$H^1_{1n}(x) = T_n(x), \qquad n = 0,1,2,3\dots \tag{12}$$

where $T_n(x)$ is the Chebyshev polynomials, indeed a first kind of Bachok-Hasham polynomial of order $k, H^k_{1n}(x)$ is a function of Chebyshev polynomials in x^k, so that we state the relation between the Chebyshev polynomials and the Bachok-Hasham polynomial of order k,

2.4. Properties of Bachok-Hasham polynomials

Let interval and weight function be defined by

$$[a,b] = [-1,1], w(x) = \frac{x^{k-1}}{\sqrt{1-x^{2k}}}, \text{k is positive odd integer},$$

then the inner product Bachok-Hasham polynomials of first kind $H^k_{1n}(x)$ is defined as

$$\langle H^k_{1n},H^k_{1m}\rangle = \int_{-1}^{1}\frac{x^{k-1}H^k_{1n}(x)H^k_{1m}(x)}{\sqrt{1-x^{2k}}}dx.$$

$n,m = 0,1,2,3,\dots$

Lemma 2.2. *Let $H^k_{1n}(x)$ be a Bachok-Hasham polynomials of the first kind then*

(i) $H^k_{1n}(x) = T_n(x^k) = \cos(n\cos^{-1}(x^k)), \quad k = 1,3,5,\dots; \quad n = 0,1,2,\dots,$ where $T_n(x)$ be Chebyshev polynomials.

(ii) The set of first kind Bachok-Hasham polynomials $H^k_{1n}(x), k = 2l-1, n = 0, 1, 2, \ldots$ is a set of orthogonal function over $[-1, 1]$, with respect to the weight function $w(x) = \frac{x^{k-1}}{\sqrt{1-x^{2k}}}$

$$\langle H^k_{1n}, H^k_{1m}\rangle = \int_{-1}^{1} \frac{x^{k-1} H^k_{1n}(x) H^k_{1m}(x)}{\sqrt{1-x^{2k}}}dx = \begin{cases} 0, & m \neq n, \\ \frac{\pi}{2k}, & m = n, \\ \frac{\pi}{k}, & m = n = 0. \end{cases} \tag{13}$$

Proof.

(i) The proof directly comes by letting $u = x^k$.

(ii) Using the relations $H^k_{1n}(x) = \cos(n\cos^{-1}(x^k))$ and $H^k_{1m}(x) = \cos(m\cos^{-1}(x^k))$, for $n, m = \{0, 1, 2, 3, \ldots\}$ and letting $u = x^k \Rightarrow du = kx^{k-1}dx$, we obtain,

$$\begin{aligned} \langle H^k_{1n}, H^k_{1m}\rangle &= \int_{-1}^{1} \frac{x^{k-1} H^k_{1n}(x) H^k_{1m}(x)}{\sqrt{1-x^{2k}}}dx \\ &= \int_{-1}^{1} \frac{x^{k-1}\cos(n\cos^{-1}(x^k))\cos(m\cos^{-1}(x^k))}{\sqrt{1-x^{2k}}}dx \\ &= \frac{1}{k}\int_{-1}^{1} \frac{\cos(n\cos^{-1}(u))\cos(m\cos^{-1}(u))}{\sqrt{1-u^2}}du \\ &= \frac{1}{k}\int_0^{\pi} \frac{\cos(n\theta)\cos(m\theta)\sin(\theta)}{\sqrt{1-\cos^2(\theta)}}d\theta \\ &= \frac{1}{k}\int_0^{\pi} \cos(n\theta)\cos(m\theta)d\theta. \end{aligned} \tag{14}$$

Hence

$$\langle H_{1n}, H_{1m}\rangle = 0, (n \neq m) \tag{15}$$

The norm of $H^k_{1n}(x)$ is

$$\begin{aligned} \|H^k_{1n}\|^2 &= \langle H^k_{1n}, H^k_{1n}\rangle = \int_{-1}^{1} \frac{x^{k-1}\left[H^k_{1n}(x)\right]^2}{\sqrt{1-x^{2k}}}dx \\ &= \frac{1}{k}\int_0^{\pi} \cos^2(n\theta)d\theta \\ &= \frac{\pi}{2k}, \quad n = 1, 2, \cdots \end{aligned} \tag{16}$$

while, $n = 0$

$$\|H^k_{10}\|^2 = \langle H^k_{10}, H^k_{10}\rangle = \frac{\pi}{k}. \quad \square$$

Lemma 2.3 (*Berthold et al. [28]*).

$$\int_0^{\pi} \frac{\cos(n\theta)}{\cos(\theta) - \cos(\phi)}d\theta = \pi\frac{\sin(n\phi)}{\sin(\phi)}, \tag{17}$$

for any $\phi \in [0, \pi]$, $n = 1, 2, 3, \ldots$

Proof. Even if inequality (17) is known prove of it is not given in details, therefore we sketch the prove in details. Its easy to show that

$$\begin{aligned} I_n(\phi) &= \int_0^{\pi} \frac{\cos(n\theta)}{\cos(\theta) - \cos(\phi)}d\theta \\ &= \frac{1}{2}\int_{-\pi}^{\pi} \frac{\cos(n\theta)}{\cos(\theta) - \cos(\phi)}d\theta, \quad \phi \in [0, \pi]. \end{aligned} \tag{18}$$

It is known that the $\cos(\theta)$ related with a complex variable z in the form

$$\cos(\theta) = \frac{1}{2}\left(z + \frac{1}{z}\right), \tag{19}$$

where $z = re^{i\theta}$ with $|z| = r = 1$.

If z moves on the unit circle, centrad at the origin, $\cos(\theta)$ traverses the interval $[-1, 1]$ twice. Using Euler formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta), \tag{20}$$

we obtain

$$z = re^{i\theta} \Rightarrow d\theta = \frac{dz}{iz} \tag{21}$$

$$\cos(n\theta) = \frac{e^{in\theta} + e^{-in\theta}}{2} = \frac{z^n + \frac{1}{z^n}}{2}. \tag{22}$$

Substituting Eqs. (19)–(22) into Eq. (18) yields

$$\begin{aligned} I_n(\phi) &= \frac{1}{2}\oint_{|z|=1} \frac{\frac{1}{2}\left(z^n + \frac{1}{z^n}\right)}{\frac{1}{2}\left(z + \frac{1}{z}\right) - \cos(\phi)}\frac{dz}{iz} = \frac{1}{2i}\oint_{|z|=1} \frac{z^{2n}+1}{z^n(z^2 - 2\cos(\phi)z + 1)}dz \\ &= \frac{1}{2i}\oint_{|z|=1} \frac{z^n}{(z-e^{i\phi})(z-e^{-i\phi})}dz + \frac{1}{2i}\oint_{|z|=1} \frac{1}{z^n(z-e^{i\phi})(z-e^{-i\phi})}dz \\ &= I_{1n}(\phi) + I_{2n}(\phi), \end{aligned} \tag{23}$$

where

$$I_{1n}(\phi) = \frac{1}{2i}\oint_{|z|=1} \frac{z^n}{(z-e^{i\phi})(z-e^{-i\phi})}dz, \tag{24}$$

$$I_{2n}(\phi) = \frac{1}{2i}\oint_{|z|=1} \frac{1}{z^n(z-e^{i\phi})(z-e^{-i\phi})}dz. \tag{25}$$

Thus, $I_n(\phi)$ in (23) is divided into two parts:

- For the first part $I_{1n}(\phi)$ of right hand side of Eq. (24), the function

$$g(z) = \frac{z^n}{(z-e^{i\phi})(z-e^{-i\phi})}, \tag{26}$$

has two simple poles of order one ($z_1 = e^{i\phi}$, and $z_2 = e^{-i\phi}$). Using Theorem 2.1, we obtain

$$Res(g(z), z_1 = e^{i\phi}) = \frac{e^{in\phi}}{e^{i\phi} - e^{-i\phi}}, \tag{27}$$

$$Res(g(z), z_2 = e^{-i\phi}) = -\frac{e^{-in\phi}}{e^{i\phi} - e^{-i\phi}}. \tag{28}$$

Substitute (27) and (28) into (24) to get

$$I_{1n}(\phi) = \frac{1}{2i}\oint_{|z|=1} \frac{z^n}{(z-e^{i\phi})(z-e^{-i\phi})}dz = \pi\frac{\sin(n\phi)}{\sin(\phi)}. \tag{29}$$

- For the second part $I_{2n}(\phi)$ of Eq. (25), the function

$$f(z) = \frac{1}{z^n(z-e^{i\phi})(z-e^{-i\phi})}, \tag{30}$$

has three poles, two of them of order one ($z_1 = e^{i\phi}, z_2 = e^{-i\phi}$), and other pole at $z = 0$ with order n, the residue of $f(z)$ at $z_1 = e^{i\phi}$ and $z_2 = e^{-i\phi}$ are

$$Res(f(z), z_1 = e^{i\phi}) = \frac{e^{-in\phi}}{(e^{i\phi} - e^{-i\phi})}, \tag{31}$$

$$Res(f(z), z_2 = e^{-i\phi}) = \frac{e^{in\phi}}{(e^{-i\phi} - e^{i\phi})}. \tag{32}$$

and the residue of $f(z)$ at $z_3 = 0$, with order n is

$$\begin{aligned} Res(f(z), z_3 = 0) &= \lim_{x \to 0} \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\left[(z-0)^n \frac{1}{z^n(z-e^{i\phi})(z-e^{-i\phi})}]\right] \\ &= \frac{1}{(n-1)!}\frac{1}{2i\sin(\phi)}\left[\frac{(-1)^{n-1}(n-1)!}{(-1)^n e^{in\phi}} - \frac{(-1)^{n-1}(n-1)!}{(-1)^n e^{-in\phi}}\right] \\ &= \frac{1}{2i\sin(\phi)}\left[\frac{-1}{e^{in\phi}} + \frac{1}{e^{-in\phi}}\right] = \frac{\sin(n\phi)}{\sin(\phi)}. \end{aligned} \tag{33}$$

Substituting Eqs. (32)–(34) into (25) yields

$$\begin{aligned} I_{2n}(\phi) &= \tfrac{1}{2i} 2\pi i \left[Res(f(z), z_1 = e^{i\phi}) + Res(f(z), z_1 = e^{-i\phi}) + Res(f(z), z_0 = 0) \right] \\ &= \pi \left[\tfrac{e^{-in\phi}}{e^{in\phi} - e^{-in\phi}} + \tfrac{e^{in\phi}}{e^{-in\phi} - e^{in\phi}} + \tfrac{\sin n\phi}{\sin \phi} \right] = 0. \end{aligned} \tag{34}$$

The proof of Lemma 2.4 can be completed by substituting (29) and (34) into (23).

□

Lemma 2.4.

$$\int_0^\pi \frac{\sin(n\theta)\sin(\theta)}{\cos(\theta) - \cos(\phi)} d\theta = -\pi \cos(n\phi),$$

for any ϕ in $[0, \pi]$, n = 1, 2, 3,…

Theorem 2.5. *The integral equation (Berthold et al. [28])*

$$\lambda \phi(t) = \int_{-1}^{1} \frac{1}{\sqrt{1 - x^2}} \phi(x) \log|x - t| dx$$

has an eigensolutions $\phi(t) = \phi_n(t) = T_n(t)$ corresponding eigenvalues $\lambda = \lambda_n = \frac{\pi}{n}, n = 1, 2, 3, \cdots$ where $T_n(t)$ is Chebyshev polynomials of first kind.

Now, we state the theorem that related with first kind of Bachok-Hasham polynomials $H^k_{1n}(x)$.

Theorem 2.6. *The integral equation*

$$\lambda \phi(t) = \int_{-1}^{1} \frac{x^{k-1}}{\sqrt{1 - x^{2k}}} \phi(x) \log|x^k - t^k| dx$$

has an eigensolutions $\phi(t) = \phi_n(t) = H^k_{1n}(t)$ corresponding eigenvalues $\lambda = \lambda_n = -\frac{\pi}{nk}, n = 1, 2, 3, \cdots$ where $H^k_{1n}(t)$ is Bachok-Hasham polynomials of first kind, k is positive odd integer.

Proof. Using the definition Bachok-Hasham polynomials of first kind,

$$H^k_{1n}(x) = \cos(n\theta) = \cos(n \cos^{-1}(x^k)), \quad x^k = \cos(\theta),$$

and applying Theorem 2.6 together with replacing $\phi(x) = H^k_{1n}(x) = \cos(n \cos^{-1}(x^k))$ and letting $u = x^k \Rightarrow du = kx^{k-1} dx$with $x = -1 \Rightarrow u = -1,$ and $x = 1 \Rightarrow u = 1,$ we obtain

$$\begin{aligned} I^*_n(\phi) = &\int_{-1}^{1} \tfrac{x^{k-1}}{\sqrt{1-x^{2k}}} H^k_{1n}(x) \log|x^k - t^k| dx \\ = &\int_{-1}^{1} \tfrac{x^{k-1}}{\sqrt{1-x^{2k}}} \cos(n \cos^{-1}(x^k)) \log|x^k - t^k| dx \\ = &\tfrac{1}{k} \int_{-1}^{1} \tfrac{\cos n(\cos^{-1}(u))}{\sqrt{1-u^2}} \log|u - t^k| du \\ = &\tfrac{1}{k} \int_{-1}^{1} \tfrac{T_n(u)}{\sqrt{1-u^2}} \log|u - t^k| du. \\ = &-\tfrac{\pi}{nk} T_n(t^k) \\ = &-\tfrac{\pi}{nk} H^k_{1n}(t). \end{aligned} \tag{35}$$

where $n = 1, 2, 3, \ldots; k$ is positive odd integer. It implies the proof of the Theorem 2.6. □

3. Function approximation

In this section we will investigate first and second kind (homogenies and inhomogeneous) of logarithmic singular integral equation (LogSIEs, Eqs. (1) and (2)). Without loss of generality we can assume that $\lambda = 1$, main technique is dealing with logarithmic kernels and reduce the integral equation problem into algebraic equations where involving matrices. The key point is to approximate the unknown function as finite series of the Bachok–Hasham polynomial $H^k_{1i}(t)$.

3.1. The logarithmic singular integral equation of first kind

Let us consider the linear LogSIEs (1) of first kind

$$g(t) = \int_{-1}^{1} L(x, t) f(x) dx, \qquad x, t \in [-1, 1] \tag{36}$$

where the left hand side $g(t)$ is given by

$$L(x, t) = \log|x^k - t^k|, kispositiveoddinteger \tag{37}$$

and need to be determined $f(t)$.

Proceeding in the usual way, we approximate f by:

$$f_n(t) = a_0 H^k_0(t) \frac{t^{k-1}}{\sqrt{1 - t^{2k}}} + \frac{t^{k-1}}{\sqrt{1 - t^{2k}}} \sum_{i=1}^{n} a_i H^k_{1i}(t),$$

$$\text{k is positive odd integer} \tag{38}$$

Substitute Eq. (38) into Eq. (36) to get:

$$g(t) = a_0 \langle L, H^k_{10} \rangle_{w_k} + \sum_{i=1}^{n} a_i \langle L, H^k_{1i} \rangle_{w_k} \tag{39}$$

Orthogonality property of the Bachok–Hasham polynomials of first kind $H^k_{1i}(x)$ Eq. (13) yields:

$$g(t) = -a_0 \frac{\pi}{k} \log 2 - \pi \sum_{i=1}^{n} a_i \frac{H^k_{1i}(t)}{i \times k} \tag{40}$$

where the coefficients a_i are unknown to be determined.

To determine the unknown coefficients a_i we use the classical Galerkin method. To get $(n + 1)$ unknowns and same number of equations we multiply both sides of Eq. (40) by $H^k_{1j}(t), j = 0, 1, 2, \ldots, n$, then integrating over the interval $[-1, 1]$ and using Theorem 2.3, Eq. (40) becomes:

$$\langle g, H^k_{1j} \rangle = -a_0 \frac{\pi}{k} \log 2 \langle 1, H^k_{1j} \rangle - \pi \sum_{i=1}^{n} \frac{a_i}{i \times k} \langle H^k_{1i}, H^k_{1j} \rangle, \tag{41}$$

where $j = 0, 1, 2, \ldots, n$, Eq. (41) can be simplified as

$$a_0 \pi \frac{1}{k} \log 2\beta^k_i + \pi \sum_{i=1}^{n} a_i \frac{1}{i \times k} \langle H^k_{1i}, H^k_{1j} \rangle = -\langle g, H^k_{1j} \rangle \tag{42}$$

The linear simultaneous equations generated from Eq. (42) formulate a system of linear algebraic equations which can be represented as a matrix form:

$$\begin{bmatrix} \frac{\pi}{k} \log 2\beta^k_0 & \pi \frac{\alpha^k_{0,1}}{1 \times k} & \pi \frac{\alpha^k_{0,2}}{2 \times k} & \pi \frac{\alpha^k_{0,3}}{3 \times k} & \cdots & \pi \frac{\alpha^k_{0,n}}{n \times k} \\ \frac{\pi}{k} \log 2\beta^k_1 & \pi \frac{\alpha^k_{1,1}}{1 \times k} & \pi \frac{\alpha^k_{1,2}}{2 \times k} & \pi \frac{\alpha^k_{1,3}}{3 \times k} & \cdots & \pi \frac{\alpha^k_{1,n}}{n \times k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\pi}{k} \log 2\beta^k_i & \pi \frac{\alpha^k_{i,1}}{i \times k} & \pi \frac{\alpha^k_{i,2}}{i \times k} & \pi \frac{\alpha^k_{i,n}}{i \times k} & \cdots & \pi \frac{\alpha^k_{i,n}}{i \times k} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\pi}{k} \log 2\beta^k_n & \pi \frac{\alpha^k_{n,1}}{n \times k} & \pi \frac{\alpha^k_{n,2}}{n \times k} & \pi \frac{\alpha^k_{n,3}}{n \times k} & \cdots & \pi \frac{\alpha^k_{n,n}}{n \times k} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_i \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} -C^k_0 \\ -C^k_1 \\ \vdots \\ -C^k_i \\ \vdots \\ -C^k_n \end{bmatrix} \tag{43}$$

where,

$$\begin{aligned} &\beta_i^k = \int_{-1}^{1} H_{1i}^k(t)dt \\ &\alpha_{i,j}^k = \langle H_i^{1k}, H_{1j}^k \rangle \qquad i = 1,2,3,\ldots,n; j = 0,1,2,3,\ldots,n \\ &C_j^k = \langle g, H_{1j}^k \rangle \end{aligned} \tag{44}$$

The above matrix has size $(n+1)\times(n+1)$, we use MATLAB to determine coefficients $a_i, i = 0,1,2,3,\ldots,n$.

3.2. The logarithmic singular integral equation of second kind

3.2.1. Nonhomogeneous case

Let us rewrite (2) in the form

$$f(t) = g(t) + \int_{-1}^{1} L(x,t)f(x)dx, \qquad x,t \in [-1,1] \tag{45}$$

where the right hand side function $g(t)$ and the logarithmic kernel function $L(x,t)$ is defined as (37) are given, $f(t)$ is the unknown function to be determined.

For convenience of exposition we rewrite (45) in operator form:

$$f = g + Tf, \tag{46}$$

where

$$Tf = \int_{-1}^{1} L(x,t)f(x)dx, \qquad x,t \in [-1,1] \tag{47}$$

We approximate $f(t)$ as mentioned progress in , Substitute Eq. (38) into Eq. (46) to get:

$$a_0 w_k(t)H_{10}^k(t) + \sum_{i=1}^{n} a_i w_k(t)H_{1i}^k(t) = g(t) + \sum_{i=1}^{n} a_i \langle L, H_{1i}^k \rangle_{w_k} \tag{48}$$

Use the orthogonality property of the first kind of Bachok–Hasham polynomials $H_{1i}^k(x)$ with weight $w_k(x) = \frac{x^{k-1}}{\sqrt{1-x^{2k}}}$, Eq. (48) yields:

$$\begin{aligned} &a_0 w_k(t)H_{10}^k(t) + \sum_{i=1}^{n} a_i w_k(t)H_{1i}^k(t) \\ &\quad = g(t) - a_0 \frac{\pi}{k}\log 2 - \pi\sum_{i=1}^{n} a_i \frac{H_{1i}^k(t)}{i\times k} \end{aligned} \tag{49}$$

where the coefficients a_i are unknown to be determined. we multiply both sides of Eq. (49) by $H_{1j}^k(t), j = 0,1,2,\ldots,n$, then integrating it over the interval $[-1,1]$ and using Theorem 2.3, Eq. (49) yields:

$$\begin{aligned} &a_0 \langle H_{10}^k, H_{1j}^k \rangle_{w_k} + \sum_{i=1}^{n} a_i \langle H_{10}^k, H_{1j}^k \rangle_{w_k} \\ &\quad = \langle g, H_{1j}^k \rangle - a_0 \frac{\pi}{k}\log 2 \langle 1, H_{1j}^k \rangle - \pi\sum_{i=1}^{n} \frac{a_i}{i\times k} \langle H_{1i}^k, H_{1j}^k \rangle, \end{aligned} \tag{50}$$

where $j = 0,1,2,\ldots,n$, Eq. (50) can be written as

$$\begin{aligned} &a_0 \pi \left[\langle H_{10}^k, H_{1j}^k \rangle_{w_k} + \frac{1}{k}\log 2\beta_n^k \right] \\ &\quad + \pi\sum_{i=1}^{n} a_i \left[\langle H_{10}^k, H_{1j}^k \rangle_{w_k} + \frac{1}{i\times k} \langle H_{1i}^k, H_{1j}^k \rangle \right] \\ &\quad = \langle g, H_{1j}^k \rangle \end{aligned} \tag{51}$$

Similarly to the previous steps in nonhomogeneous case, the unknown coefficients a_i can be determine by solving the system of linear equations which can represented by a matrix form as

$$\begin{bmatrix} \frac{\pi}{k}[1+\log 2\beta_0^k] & \pi\frac{\alpha_{0,1}^k}{1\times k} & \pi\frac{\alpha_{0,2}^k}{2\times k} & \pi\frac{\alpha_{0,3}^k}{3\times k} & \cdots & \pi\frac{\alpha_{0,n}^k}{n\times k} \\ \frac{\pi}{k}\log 2\beta_1^k & \pi[\frac{1}{2\times k}+\frac{\alpha_{1,1}^k}{1\times k}] & \pi\frac{\alpha_{1,2}^k}{2\times k} & \pi\frac{\alpha_{1,3}^k}{3\times k} & \cdots & \pi\frac{\alpha_{1,n}^k}{n\times k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\pi}{k}\log 2\beta_i^k & \pi\frac{\alpha_{i,1}^k}{i\times k} & \pi\frac{\alpha_{i,2}^k}{i\times k} & \pi[\frac{1}{2\times k}+\frac{\alpha_{i,n}^k}{i\times k}] & \cdots & \pi\frac{\alpha_{i,n}^k}{i\times k} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\pi}{k}\log 2\beta_n^k & \pi\frac{\alpha_{n,1}^k}{n\times k} & \pi\frac{\alpha_{n,2}^k}{n\times k} & \pi\frac{\alpha_{n,3}^k}{n\times k} & \cdots & \pi[\frac{1}{2\times k}+\frac{\alpha_{n,n}^k}{n\times k}] \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_i \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} C_0^k \\ C_1^k \\ \vdots \\ C_i^k \\ \vdots \\ \vdots C_n^k \end{bmatrix} \tag{52}$$

where $\beta_i^k, \alpha_{i,n}^k, C_i^k$ are described in Eq. (44)

4. Numerical examples

This section reports on two numerical experiments to study the accuracy and the performance of the proposed adaptive strategy method. All experiments have been conducted using MATLAB. All examples are academic in the sense that the exact solution is always known (Thereom 2.6). Two examples were performed with $n = 4$ and $n = 9$. We restrict the presentation to the simplest case $k = 3,5$ respectively. The right-hand side for the Galerkin method is always computed as explained in Section 3.

Example 1. 1 Consider a Nonhomogeneous linear LogSIEs of second kind:

$$f(t) = \frac{3t^5 + \pi t^3\sqrt{1-t^6}}{3\sqrt{1-t^6}} + \int_{-1}^{1} f(x)\log|x^3 - t^3|dx, x,t \in [-1,1] \tag{53}$$

The exact solution of Eq. (53) is $f(t) = \frac{t^5}{\sqrt{1-t^6}}$; The absolute errors between exact and approximation solutions are given by Tables 1. The results presented shows a very good convergence of the values use first kind of Bachok–Hasham polynomials.

Example 2. 2 Consider a linear LogSIEs of first kind

$$t^5 = -\frac{5}{\pi}\int_{-1}^{1} f(x)\log|x^5 - t^5|dx, x,t \in [-1,1] \tag{54}$$

The exact solution of Eq. (54) is $f(x) = \frac{x^9}{\sqrt{1-x^{10}}}$. The absolute errors between exact and approximation solutions are given by Tables 2.

Table 1
Absolute error for Example 1.

x	n = 4	n = 9
0.9	1.11022302462516 E−16	1.11022302462516 E−16
0.8	1.11022302462516 E−16	1.11022302462516 E−16
0.7	1.11022302462516 E−16	8.32667268468867 E−17
0.6	0	1.38777878078145 E−17
0.5	2.08166817117217 E−17	2.77555756156289 E−17
0.4	6.93889390390723 E−18	5.20417042793042 E−18
0.3	5.20417042793042 E−18	5.20417042793042 E−18
0.2	3.30681662608079 E−18	3.25260651745651 E−18
0.1	1.10453096321961 E−18	1.10114283143059 E−18
0	0	0
−0.1	1.08420217248550 E−19	1.06726151354042 E−19
−0.2	7.04731412115578 E−19	7.58941520739853 E−19
−0.3	6.07153216591883 E−18	5.63785129692462 E−18
−0.4	8.67361737988404 E−18	1.04083408558608 E−17
−0.5	6.93889390390723 E−18	0
−0.6	4.16333634234434 E−17	4.16333634234434 E−17
−0.7	5.55111512312578 E−17	2.77555756156289 E−17
−0.8	1.11022302462516 E−16	1.11022302462516 E−16
−0.9	2.22044604925031 E−16	2.22044604925031 E−16

Table 2
Absolute error for Example 2

x	n = 4	n = 9
0.9	5.55111512312578 E−17	5.55111512312578 E−17
0.8	2.77555756156289 E−17	4.99600361081320 E−16
0.7	1.38777878078145 E−17	2.08166817117217 E−16
0.6	1.73472347597681 E−18	6.07153216591883 E−17
0.5	8.67361737988404 E−19	1.17093834628434 E−17
0.4	5.42101086242752 E−19	2.05998412772246 E−18
0.3	6.26804380968182 E−19	5.14996031930615 E−19
0.2	2.03287907341032 E−20	2.31875269310865 E−20
0.1	3.99218043133405 E−21	3.98659696219932 E−21
0	0	0
−0.1	5.96562657773244 E−21	5.97121004686717 E−21
−0.2	1.75653457436861 E−19	1.72794721239877 E−19
−0.3	1.82959116606929 E−19	2.94767465644497 E−19
−0.4	2.60208521396521 E−18	1.13841228110978 E−18
−0.5	6.93889390390723 E−18	1.77809156287623 E−17
−0.6	1.38777878078145 E−17	4.33680868994202 E−17
−0.7	6.93889390390723 E−18	2.28983498828939 E−16
−0.8	2.77555756156289 E−17	4.44089209850063 E−16
−0.9	5.55111512312578 E−17	5.55111512312578 E−17

5. Conclusion

We have introduced a new class of orthogonal polynomials $H_{1n}^{k}(t)$ (named Bachok-Hasham polynomials of first kind) with weight functions $w_k(x) = \frac{x^{k-1}}{\sqrt{1-x^{2k}}}$ and used it to find approximate solution of a logarithmic singular integral equations of first and second kind Eqs. (1) and (2). It is proven (Theorem 2.6) that Bachok-Hasham polynomials of first kind is a eigenfunction of the homogeneous LogSIE of the second kind corresponding eigenvalues $\lambda = \lambda_n = -\frac{\pi}{nk}, n = 1, 2, ..,$ Numerical results (Example 1 and Example 2) show that Bachok-Hasham polynomials of first kind is very effective and accurate for the problems stated above.

Acknowledgments

This work was supported by Universiti Sains Islam Malaysia (USIM) under Research Grand PPP/GP/FST/30/14915. Authors are grateful for sponsor and financial support of the Research Management Center (RMC) of USIM.

References

[1] Corduneanu C. Integral equations and applications, vol. 148. Cambridge: Cambridge University Press; 1991.
[2] Polyanin AD, Manzhirov AV. Handbook of integral equations. CRC Press; 2008.
[3] Porter D, Stirling DS. Integral equations: a practical treatment, from spectral theory to applications, vol. 5. Cambridge University Press; 1990.
[4] Hochstadt H. Integral equations, vol. 91. John Wiley and Sons; 2011.
[5] Hsiao CH, Chen CF. Solving integral equations via Walsh functions. Comput Electrical Eng 1979;6(4):279–92.
[6] Nosrati Sahlan M, Marasi HR, Ghahramani F. Block-pulse functions approach to numerical solution of Abel's integral equation. Cogent Math 2015;2(1):1047111.
[7] Jafarian A, Esmailzadeh Z. On the numerical solution of Urysohn integral equation using Legendre approximation. J Math Model 2013;1:76–84.
[8] Aboiyar T, Kuhe DA, Soomiyol MC. On Laguerre-Gaussians and Fredholm integral equations. Afr J Math Comput Sci Res 2010;3(10):220–4.
[9] Paryab K, Rostami M. Expansion method for linear Fredholm integral equations of second kind by Chebyshev, Legendre and Shannon wavelats and the comparison of their numerical results. Math Sci Quart J 2008;2(4):335–46.
[10] Ervin VJ, Stephan EP. Collocation with Chebyshev polynomials for a hypersingular integral equation on an interval. J Comput Appl Math 1992;43(1):221–9.
[11] Iovane G, Ciarletta M. Hypersingular integral equations and applications to porous elastic materials; 2003. arXiv preprint math-ph/0302055.
[12] Chan YS, Fannjiang AC, Paulino GH. Integral equations with hypersingular kernels, theory and applications to fracture mechanics. Int J Eng Sci 2003;41(7):683–720.
[13] Mahiub MA, Nik Long NMA, Eshkuvatov ZK. Numerical solution of hypersingular integral equation. Int J Pure Appl Math 2011;69(3):265–74.
[14] Eshkuvatov ZK, Nik Long NMA, Abdulkawi M. Approximate solution of singular integral equations of the first kind with Cauchy kernel. Appl Math Lett 2009;22(5):651–7.
[15] Sizikov V, Sidorov D. Generalized quadrature for solving singular integral equations of Abel type in application to infrared tomography; 2015. arXiv preprint arXiv:1509.02586.
[16] Chakrabarti A, Manam SR. Solution of a logarithmic singular integral equation. Appl Math Lett 2003;16(3):369–73.
[17] Obaiys Suzan J, Eskhuvatov ZK, Nik long NMA, Jamaludin MA. Galerkin method for the numerical solution of hypersingular integral equations based chebyshev polynomials. Int J Math Anal 2012; 6(54):2653–64.
[18] Dardery SM, Allan MM. Chebyshev polynomials for solving a class of singular integral equations. Appl Math 2014;5(04):753.
[19] Jiang S, Rokhlin V. Second kind integral equations for the classical potential theory on open surfaces II. J Comput Phys 2004;195(1):1–16.
[20] Chakrabarti A, Sahoo T. Solution of singular integral equations with logarithmic and Cauchy kernels. In: Proceedings of the Indian academy of sciences-mathematical sciences, vol. 106, No. 3. Springer India; 1996. p. 261–70.
[21] Banerjea S, Sarkar C. On a singular integral equation with log kernel and its application. Int J Math Math Sci; 2006.
[22] Kabir H, Madenci E, Ortega A. Numerical solution of integral equations with logarithmic, Cauchy and Hadamard type singularities. Int J Numer Methods Eng 1998;41(4):617–38.
[23] Pritsker I. How to find a measure from its potential. Comput Methods Function Theory 2008;8(2):597–614.
[24] Nadir M. Adapted quadratic approximation for logarithmic kernel integrals. Fasciculi mathematici 2012;49(2):75.
[25] Boyd JP. Chebyshev and Fourier spectral methods. Courier Corporation; 2001.
[26] Spiegel MR. Laplace transforms. New York: McGraw-Hill; 1965. p. 46.
[27] Mason JC, Handscomb DC. Chebyshev polynomials. CRC Press; 2002.
[28] Berthold D, Hoppe W, Silbermann B. The numerical solution of the generalized airfoil equation. J Integral Equ Appl 1992;4(2):309–36.
[29] Muskhelishvili NI. Singular integral equations.(Russian) Boundary value problems in the theory of function and some applications of them to mathematical physics. Third, corrected and augmented edition. With an appendix by B. Bojarski. Izdat.“Nauka, Moscow; 1968.
[30] Nadir M, Antidze J. On the numerical solution of singular integral equations using Sanikidze's approximation. Comp Methods Sci Technol 2004;10:83–9.
[31] Southwell RV, Vaisey G. Relaxation methods applied to engineering problems. XII. Fluid motions characterized by'free'stream-lines. Philos Trans Roy Soc London A: Math Phys Eng Sci 1946;240(815):117–61.
[32] Liggett JA. Location of free surface in porous media. J Hydraulics Division 1977;103(4):353–65.