

CHAPTER 4

EXTENDED CHEBYSHEV POLYNOMIALS OF THE THIRD AND FOURTH KINDS OF ORDER k

4.1 Introduction

This chapter is devoted to define a new class of orthogonal polynomials named Extended Chebyshev polynomials of the third and fourth kinds of order k , denoted as $Z_{(i,n)}^k(x)$, $i = \{3, 4\}$, k is positive odd integers, $n = 0, 1, 2, \dots$, which is an extension of the Chebyshev polynomials of the third and fourth kinds respectively. We will provide some elementary formulae for the recurrence relations of ECPs and summarize the key properties which will be developed. Some of the applications will be introduced and discussed. The zeros of the ECPs of the third and fourth kinds of order k are discussed and described. The main characteristics of Extended Chebyshev sequences $\{Z_{(i,n)}^{2m+1}(x)\}_{m=0}^k$, $i = \{1, 2\}$; $n = 0, 1, 2, 3, \dots$ are described when k is sufficient large odd integers. It was found that Extended Chebyshev sequences of the third and fourth kinds converge approximately into one of two values $\{-1, 1\}$ depending on index n .

Recurrence relations and the relationships between each other are proved. We state some of the associated relations of the four kinds.

It was found that ECPs of the third and fourth kinds $Z_{(i,n)}^k(x)$, $i = \{3, 4\}$ are orthogonal with respect to weight functions, $w_{(3,k)}(x) = x^{k-1} \sqrt{\frac{1+x^k}{1-x^k}}$ and $w_{(4,k)}(x) = x^{k-1} \sqrt{\frac{1-x^k}{1+x^k}}$ respectively, on the interval $[-1, 1]$, where k is positive odd integers. Spectral properties ECPs of the third and fourth kinds $Z_{(i,n)}^k(x)$, $i = \{3, 4\}$ are proved. They will be used in greater detail in Chapter 4. Moving into the applications of ECPs of the third and fourth kinds of order k , we also demonstrate how to use these orthogonal polynomials to aid in the solution of SSIE $_k$. The unknown functions are characterized in terms of finite series of ECPs. Effective numerical methods are derived to evaluate the unknown functions. Furthermore, a study of effective and accurate approximate methods for solving the SSIE $_k$ of

the first kind with third and fourth cases over a finite interval $[-1, 1]$ is presented.

4.2 Definitions of Extended Chebyshev Polynomials of The Third and Fourth kinds and Their Properties

The two other families of polynomials $Z_{(3,n)}^k(x)$, $Z_{(4,n)}^k(x)$, k is positive odd integers, are constructed, which are related to $Z_{(1,n)}^k(x)$, $Z_{(2,n)}^k(x)$, but which have trigonometric definitions involving the half angle $\theta/2$ (where $x^k = \cos$ as in chapter 3).

Let us first consider two definitions of the ECPs of the third and fourth kinds of order k $Z_{(i,n)}^k(x)$, $i = \{3, 4\}$ and some properties of them.

Definition 11 *Extended Chebyshev polynomials of the third kind of order k , $Z_{(3,n)}^k(x)$ where k is odd positive integer, is a polynomial in x of degree (kn) , defined by the relation (for fixed k):*

$$Z_{(3,n)}^k(x) = \frac{\cos((n + \frac{1}{2})\theta)}{\cos \frac{1}{2}\theta}, \quad (4.1)$$

$n = 0, 1, 2, \dots$, when $x^k = \cos(\theta)$, k is positive odd integers.

Definition 12 *Extended Chebyshev polynomials of the fourth kind of order k , $Z_{(4,n)}^k(x)$ where k is odd positive integer, is a polynomial in x of degree (kn) , defined by the relation (for fixed k):*

$$Z_{(4,n)}^k(x) = \frac{\sin((n + \frac{1}{2})\theta)}{\sin \frac{1}{2}\theta}, \quad (4.2)$$

$n = 0, 1, 2, \dots$, when $x^k = \cos(\theta)$, k is positive odd integers.

The range of the variables x and θ is the same as for $Z_{(i,n)}^k(x)$, $i = \{1, 2\}$.

The relationship between Extended Chebyshev and Chebyshev polynomials is:

$$Z_{(i,n)}^k(x) = \begin{cases} V_n(x^k), & i = 3, \\ W_n(x^k), & i = 4, \end{cases} \quad (4.3)$$

where $V_n(x)$ and $W_n(x)$ are Chebyshev polynomials of the third and fourth kinds respectively, k is positive odd integer, $n = 0, 1, 2, 3, \dots$

Moreover, for $k = 1$, we have:

$$Z^1_{(i,n)}(x) = \begin{cases} V_n(x), & i = 3, \\ W_n(x), & i = 4, \end{cases} \tag{4.4}$$

where $V_n(x)$ and $W_n(x)$ are Chebyshev polynomials of the third and fourth kinds respectively, $n = 0, 1, 2, 3, \dots$

Table (4.1) refers to the first few of Extended Chebyshev polynomials of the third and fourth kinds of order k , $Z^k_{(i,n)}(x), i = \{3, 4\}, n = 0, 1, 2, \dots$ respectively.

Table 4.1: Extended Chebyshev polynomials of the third and fourth kinds

n	$Z^k_{(3,n)}(x) = V(x^k)$	$Z^k_{(4,n)}(x) = W(x^k)$
0	1	1
1	$2x^k - 1$	$2x^k + 1$
2	$4x^{2k} - 2x^k - 1$	$4x^{2k} + 2x^k - 1$
3	$8x^{3k} - 4x^{2k} - 4x^k + 1$	$8x^{3k} + 4x^{2k} - 4x^k - 1$
4	$16x^{4k} - 8x^{3k} - 12x^{2k} + 4x^k + 1$	$16x^{4k} + 8x^{3k} - 12x^{2k} - 4x^k + 1$
5	$32x^{5k} - 16x^{4k} - 32x^{3k} + 12x^{2k} + 6x^k - 1$	$32x^{5k} + 16x^{4k} - 32x^{3k} - 12x^{2k} + 6x^k + 1$

ECPs of the third and fourth kinds can be efficiently generated by the use of a recurrence relation. Indeed, these polynomials $Z^k_{(3,n)}(x)$ and $Z^k_{(4,n)}(x)$ share precisely the same recurrence relation as $Z^k_{(1,n)}(x)$ and $Z^k_{(2,n)}(x)$ and their generation differs only in the prescription of the initial condition for $n=1$, it is discussed in more details in next section.

4.2.1 Plots of the Extended Chebyshev Polynomials of the Third and Fourth kinds

In Figures 4.1 - 4.4, we show the respective graphs of the ECPs of the third kind $Z^k_{(3,n)}(x), n = 1, 2, 3, 4$, starting with value of -1 and finishing with a value of 1 , since when $x = 1 \implies \cos \theta = 1$, this means $\theta = 0 \implies \cos(n + \frac{1}{2})0 = 1$ that yields $Z^k_{(3,n)}(1) = 1$ for any $n = 0, 1, 2, 3, \dots$. The ECPs of the third kind have precisely n zeros in interval $(-1, 1)$, from the definition of ECPs of the third kind (Definition 11), the zeros for x in $(-1, 1)$ of $Z^k_{(3,n)}(x)$ must correspond to the zeros for θ in $(0, \pi)$ of $\frac{\cos(n+\frac{1}{2})\theta}{\cos(\frac{1}{2}\theta)}$ so that:

$$V(x^k) = 0, \quad (4.5)$$

$$\cos\left(n + \frac{1}{2}\right)\theta = 0, \quad (4.6)$$

We have the zeros of $Z_{(3,n)}^k(x)$

$$x = x_{(3,m)} = \sqrt[k]{\cos \frac{(m - \frac{1}{2})\pi}{n + \frac{1}{2}}}, \quad m = 1, 2, 3, \dots, n. \quad (4.7)$$

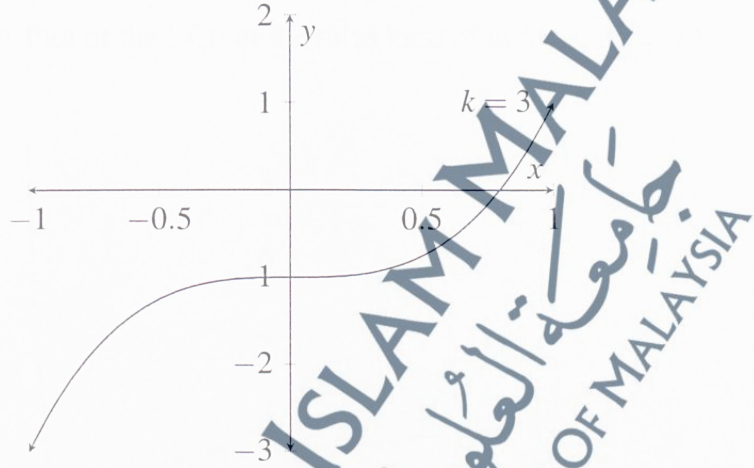


Figure 4.1: Plot of the ECP of the third kind of order 3, $Z_{(3,1)}^3(x)$



Figure 4.2: Plot of the ECP of the third kind of order 3, $Z_{(3,2)}^3(x)$

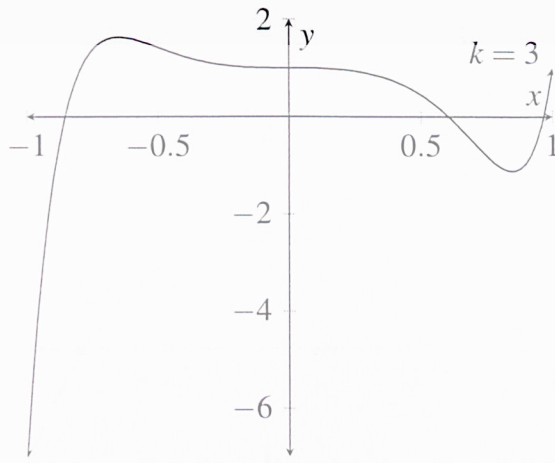


Figure 4.3: Plot of the ECP of the third kind of order 3, $Z_{(3,3)}^3(x)$

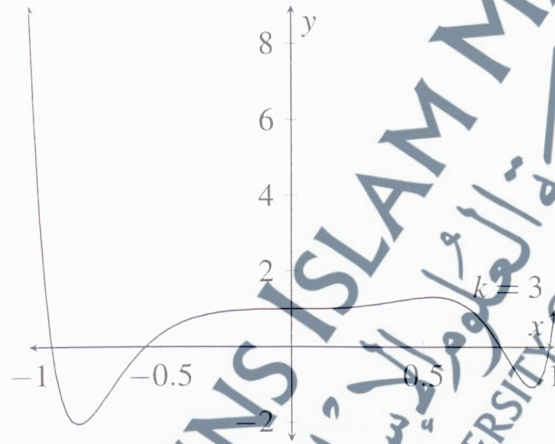


Figure 4.4: Plot of ECP of the third kind of order 3, $Z_{(3,4)}^3(x)$

The graphs of the ECPs of the fourth kind ($Z_{(4,1)}^3(x)$ to $Z_{(4,4)}^3(x)$) are plotted in Figures 4.5 - 4.8. The ECPs of the fourth kind have precisely n zeros in the interval $(-1, 1)$ (Figures 4.5 - 4.8). Moreover, the definition of ECPs of the fourth kind (definition 12) shows that the zeros for x in $(-1, 1)$ are determined in a similar way of the zeros of $\sin(n + \frac{1}{2})\theta$ as:

$$W(x^k) = 0, \quad (4.8)$$

$$x = x_{(4,m)} = \sqrt[k]{\cos\left(\frac{m}{n + \frac{1}{2}}\right)\pi}, \quad m = 1, 2, 3, \dots, n, \quad (4.9)$$

where $x_{(4,m)}$ the zeros of $Z_{(4,n)}^k(x)$.

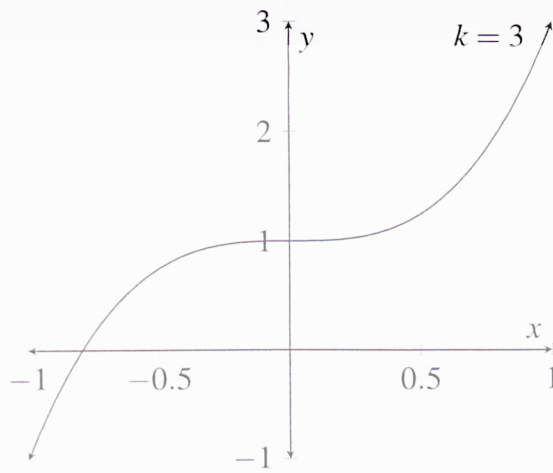


Figure 4.5: Plot of the ECP of the fourth kind of order 3, $Z_{(4,1)}^3(x)$

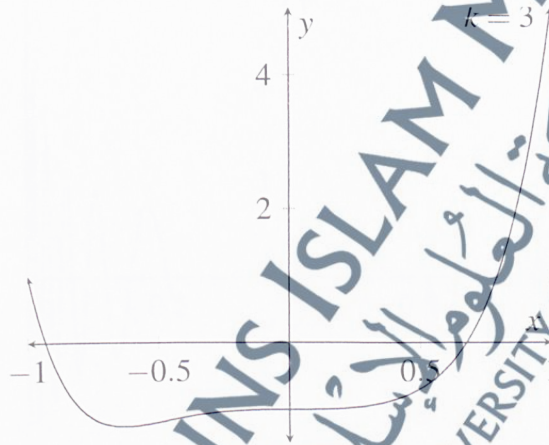


Figure 4.6: Plot of the ECP of the fourth kind of order 3, $Z_{(4,2)}^3(x)$

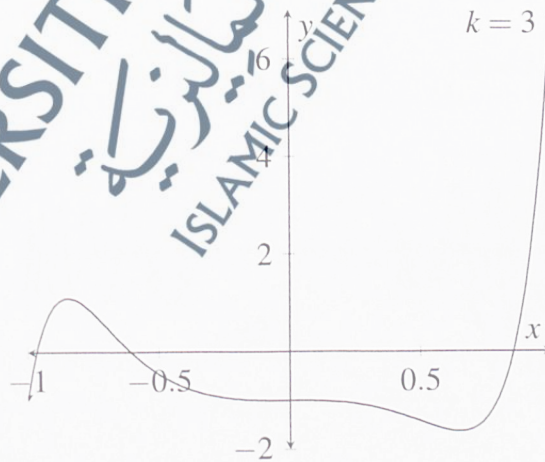


Figure 4.7: Plot of the ECPs of the fourth kind of order 3, $Z_{(4,3)}^3(x)$

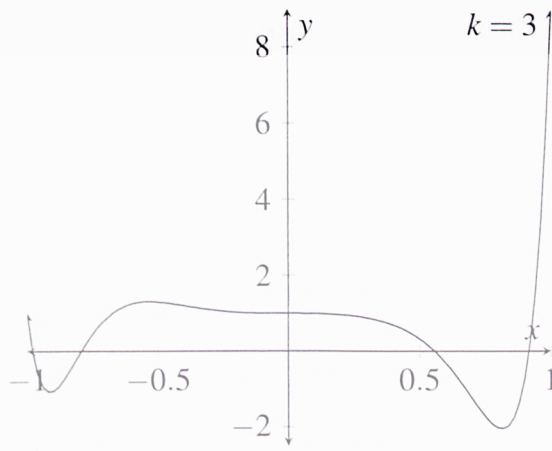


Figure 4.8: Plot of the ECP of the fourth kind of order 3, $Z_{(4,4)}^3(x)$

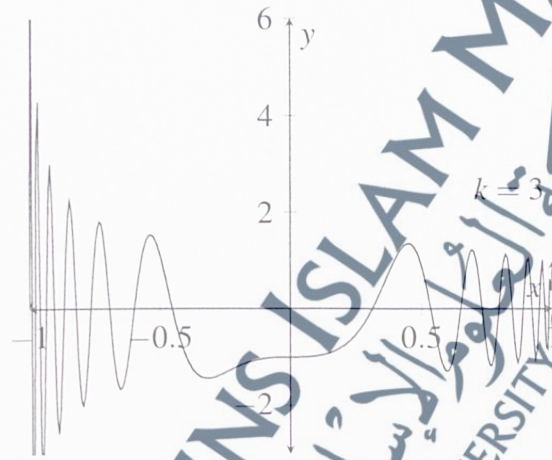


Figure 4.9: Plot of the ECP of the third kind of order 3, $Z_{(3,16)}^3(x)$

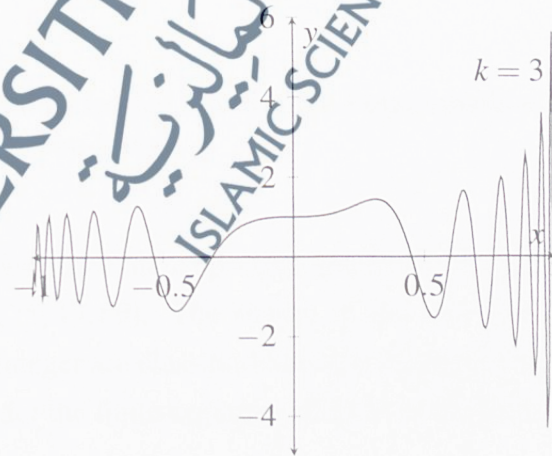


Figure 4.10: Plot of the ECP of the fourth kind of order 3, $Z_{(4,16)}^3(x)$

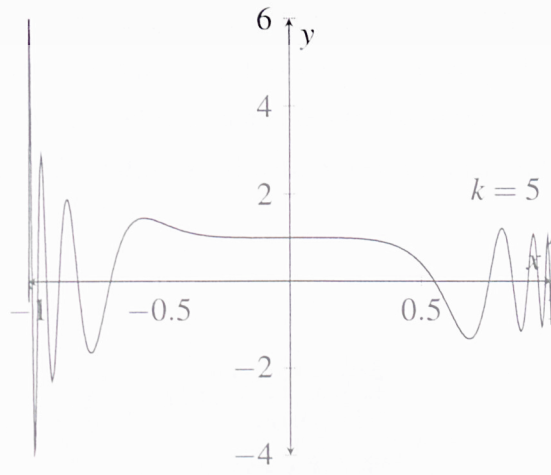


Figure 4.11: Plot of the ECP of the third kind of order 5, $Z_{(3,15)}^5(x)$



Figure 4.12: Plot of the ECP of the fourth kind of order 5, $Z_{(4,15)}^5(x)$

4.2.2 Behavior of The Extended Chebyshev Polynomials of The Third and Fourth Kinds when k is Too Large

In Figures 4.13-4.16, we show the respective graphs of $Z_{(3,n)}^k(x)$, $n = 2, 3, 4$ and different orders ($k = 1, 3, 5, 15, 25, 49, 69$). The shapes of the ECPs of the third kind when k is sufficiently large odd integer are classified into two distinct categories dependent on index n . To show that, consider the finite sequence of ECP of the third kind:

$$C = \{Z_{(3,n)}^1(x), Z_{(3,n)}^3(x), Z_{(3,n)}^5(x), \dots, Z_{(3,n)}^k(x)\}, \quad (4.10)$$

where k is large positive odd integers, $n = 0, 1, 2, 3, \dots$ and let $x \in (-1, 1)$, the term x^k becomes very small when k is sufficient too large positive odd integer. We assume that the term x^k approaches to zero which means $\cos \theta$ also approaches to zero i.e $\cos \theta \rightarrow 0 \implies \theta \rightarrow \frac{\pi}{2}$, so the ECPs of the third kind with order sufficient large k can be approximate as:

$$\begin{aligned} Z_{(3,n)}^k(x) &\approx \frac{\cos(n + \frac{1}{2})\frac{\pi}{2}}{\cos(\frac{1}{2}\frac{\pi}{2})} \\ &= \frac{\cos(n\frac{\pi}{2} + \frac{\pi}{4})}{\cos(\frac{\pi}{4})} \\ &= \cos(n\frac{\pi}{2}) - \sin(n\frac{\pi}{2}). \end{aligned} \tag{4.11}$$

The ECPs in Eq. (4.11) have approximately converged to the one of two values dependent on index n :

$$Z_{(3,n)}^k(x) \approx \begin{cases} -1, & n = 1, 2, 5, 6, \dots, \\ 1, & n = 0, 3, 4, 7, 8, \dots \end{cases} \tag{4.12}$$

where k is sufficient large positive odd integer. Figures 4.13 - 4.14 show the behavior Extended Chebyshev sequences of the third kind when $n = 1, 2$ with different values of k , $\{k = 1, 3, 5, 9, 15, 25, 35, 69\}$. The behavior of the Extended Chebyshev sequences of the third kind convergeS to -1 which coincideS to Eq . (4.12). Figures 4.15 - 4.16 show the graph of Extended Chebyshev sequences of the third kind when $n = \{3, 4\}$ with different values of k , $\{k = 1, 3, 5, 7, 15\}$, the behavior of the Extended Chebyshev sequences of the third kind converges to 1.



Figure 4.13: Plot of Extended Chebyshev sequences of the third kind for different orders k , $Z_{(3,1)}^k(x)$

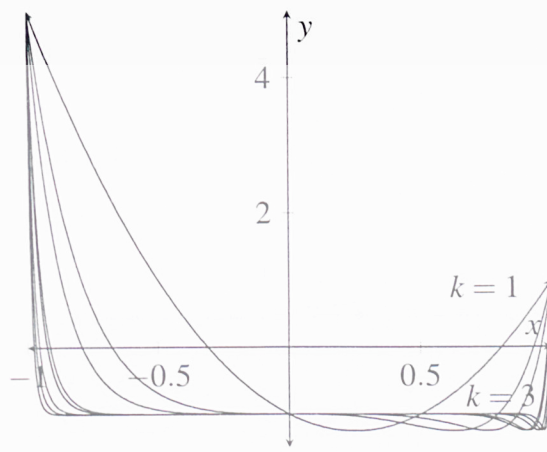


Figure 4.14: Plot of Extended Chebyshev sequences of the third kind for different orders k , $Z_{(3,2)}^k(x)$

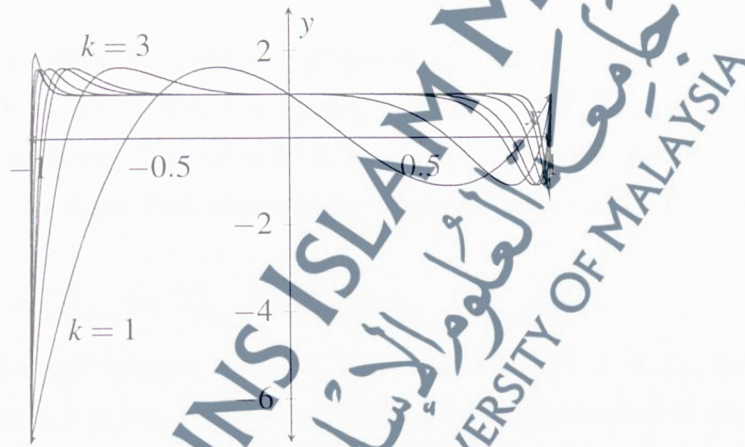


Figure 4.15: Plot of Extended Chebyshev sequences of the third kind for different orders k , $Z_{(3,3)}^k(x)$

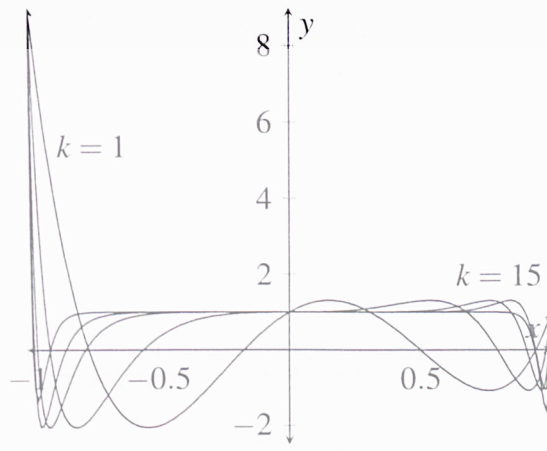


Figure 4.16: Plot of Extended Chebyshev sequences of the third kind for different orders k , $Z_{(3,4)}^3(x)$

In Figures 4.17 - 4.20, we show the respective graphs of $Z_{(1,n)}^k(x)$, $n = 1, 2, 3, 4$ with different orders of k , ($k = 1, 3, 15, 25, 35, 45$). The shapes of the Extended Chebyshev sequences of the fourth kind with sufficient large of order k are classified into two distinct categories dependent on index n . To show that, consider the finite sequence of ECP of the fourth kind:

$$D = \{Z_{(4,n)}^1(x), Z_{(4,n)}^3(x), Z_{(4,n)}^k(x), \dots, Z_{(4,n)}^k(x)\} \quad (4.13)$$

where k is large positive odd integer, $n = 0, 1, 2, 3, \dots$ and let $x \in (-1, 1)$, the term x^k becomes very small when k is too large. We assume that it approaches to zero, which means $\cos \theta$ also approaches to zero $\Rightarrow \theta \rightarrow \frac{\pi}{2}$ so the ECP of the second kind with order sufficient large k can be approximated as:

$$\begin{aligned} Z_{(4,n)}^k(x) &\approx \frac{\sin((n + \frac{1}{2})\frac{\pi}{2})}{\sin((\frac{1}{2})\frac{\pi}{2})} \\ &= \frac{\sin(\frac{n\pi}{2} + \frac{\pi}{4})}{\sin(\frac{\pi}{4})} \\ &= \cos(\frac{n\pi}{2}) + \sin(\frac{n\pi}{2}). \end{aligned} \quad (4.14)$$

The ECPs in Eq. (4.14) have approximately two values dependent on index n

$$Z_{(4,n)}^k(x) \approx \begin{cases} -1, & n = 2, 3, 6, 7, 10, 11, \dots, \\ 1, & n = 0, 1, 4, 5, 8, 9, 12, \dots \end{cases} \quad (4.15)$$

where k is sufficient large positive odd integer. Figures 4.17-4.20 show the behavior of Extended Chebyshev sequences of the fourth kind when $n = 1, 2, 3, 4$ respectively, with

different orders of k , ($k = 1, 3, 7, 11, 15, 25, 49$), the behavior of Extended Chebyshev sequences converge to $\{-1, 1\}$ which agrees with the derived formula in Eq. (4.15).

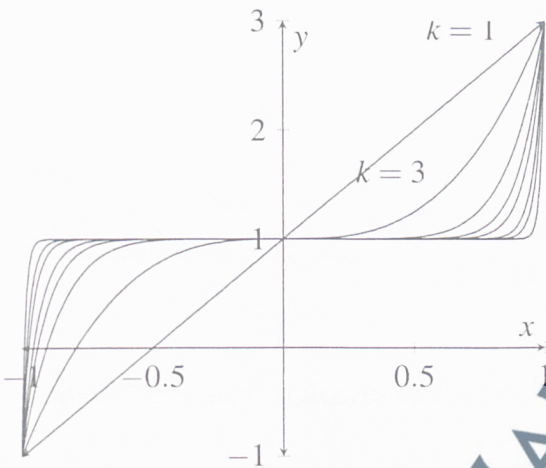


Figure 4.17: Plot of Extended Chebyshev sequences of the fourth kind of different orders k , $Z_{(4,1)}^k(x)$



Figure 4.18: Plot of Extended Chebyshev sequences of the fourth kind of different orders k , $Z_{(4,2)}^k(x)$

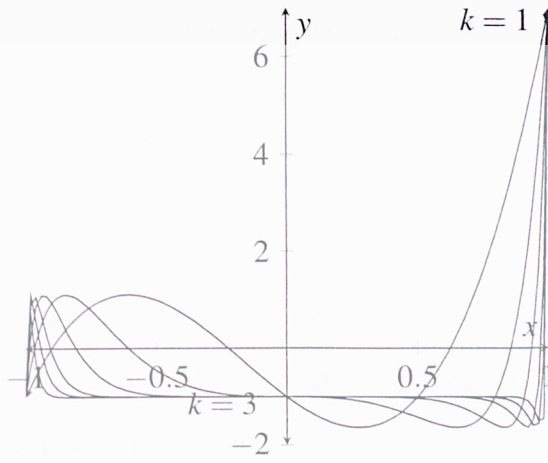


Figure 4.19: Plot of Extended Chebyshev sequences of the fourth kind of different orders k , $Z_{(4,3)}^k(x)$

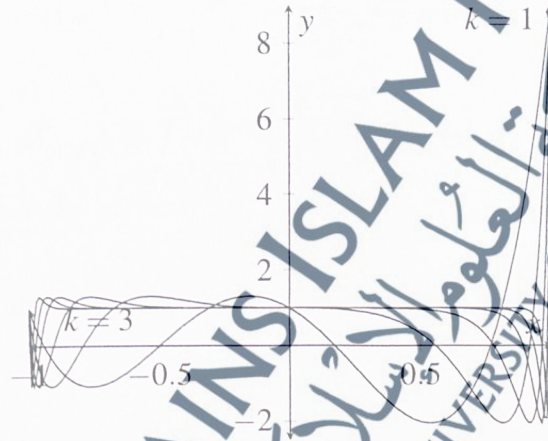


Figure 4.20: Plot of Extended Chebyshev sequences of the fourth kind for different orders k , $Z_{(4,4)}^k(x)$

4.3 Extended Chebyshev Polynomials of the Third and Fourth Kinds of Order k and Weight Functions

Let interval $[a, b] = [-1, 1]$ and weight functions be defined by:

$$w_{(i,k)}(x) = \begin{cases} x^{k-1} \frac{\sqrt{1+x^k}}{\sqrt{1-x^k}}, & i = 3, \\ x^{k-1} \frac{\sqrt{1-x^k}}{\sqrt{1+x^k}}, & i = 4, \end{cases} \quad (4.16)$$

where k is positive odd integers, then the inner product of $Z_{(i,n)}^k(x), i = \{3, 4\}$, is defined as:

$$\langle Z_{(i,n)}^k, Z_{(i,m)}^k \rangle_{w_{(i,k)}} = \begin{cases} \int_{-1}^1 w_{(3,k)}(x) Z_{(3,n)}^k Z_{(3,m)}^k dx, & i = 3, \\ \int_{-1}^1 w_{(4,k)}(x) Z_{(4,n)}^k Z_{(4,m)}^k dx, & i = 4, \end{cases} \quad (4.17)$$

k is positive odd integers, $n, m = 0, 1, 2, 3, \dots$

We found that the Extended Chebyshev polynomials of the third and fourth kinds $Z_{(i,n)}^k(x), i = \{3, 4\}$ satisfy respectively:

$$\langle Z_{(3,n)}^k, Z_{(3,m)}^k \rangle = \int_{-1}^1 x^{k-1} \frac{\sqrt{1+x^k}}{\sqrt{1-x^k}} Z_{(3,n)}^k(x) Z_{(3,m)}^k(x) dx, \quad (4.18)$$

and

$$\langle Z_{(4,n)}^k, Z_{(4,m)}^k \rangle = \int_{-1}^1 x^{k-1} \frac{\sqrt{1-x^k}}{\sqrt{1+x^k}} Z_{(4,n)}^k(x) Z_{(4,m)}^k(x) dx \quad (4.19)$$

$k = 1, 3, \dots; n, m = 0, 1, 2, 3, \dots$

Before proving the main results, we need to state this Lemma which is related to the proof:

Lemma 4.3.1 (Mason & Handscomb, 2002)

$$\int_0^\pi \frac{\cos(n + \frac{1}{2})\theta}{\cos(\theta) - \cos(\phi)} \frac{1}{2} \cos \frac{1}{2} \theta d\theta = \frac{\pi \sin(n + \frac{1}{2})\phi}{2 \sin(\frac{1}{2}\phi)}, \quad (4.20)$$

$$\int_0^\pi \frac{\sin(n + \frac{1}{2})\theta}{\cos(\theta) - \cos(\phi)} \frac{1}{2} \sin \frac{1}{2} \theta d\theta = -\frac{\pi \cos(n + \frac{1}{2})\phi}{2 \sin(\frac{1}{2}\phi)}, \quad (4.21)$$

for any ϕ in $[0, \pi]$, $n = 1, 2, 3, \dots$

Proof of Lemma 4.3.1 can be found in Mason & Handscomb (2002).

The Extended Chebyshev polynomials of the third and fourth kinds are also orthogonal systems on $[-1, 1]$, with respect to weight functions $w_{(i,k)}(x), i = \{3, 4\}$ respectively. Indeed the main results and properties can be summarized on the following Lemma:

Lemma 4.3.2 Let $Z_{(i,n)}^k(x), i = \{3, 4\}$ be the set Extended Chebyshev polynomials of the third and fourth kinds of order k , respectively and consider that the weight functions $w_{(i,k)}, i = \{3, 4\}$ be defined as in Eq. (4.16), then:

- (i) The three terms recurrence relations between Extended Chebyshev polynomials of the third and fourth kinds:

$$Z_{(3,n)}^k(x) = 2x^k Z_{(3,n-1)}^k(x) - Z_{(3,n-2)}^k(x), \quad (4.22)$$

$$Z_{(4,n)}^k(x) = 2x^k Z_{(4,n-1)}^k(x) - Z_{(4,n-2)}^k(x). \quad (4.23)$$

- (ii) Orthogonality property of the Extended Chebyshev polynomials of the third kind of order k :

$$\langle Z_{(3,n)}^k, Z_{(3,m)}^k \rangle_{w(3,k)} = \begin{cases} 0, & m \neq n, \\ \frac{\pi}{k}, & m = n. \end{cases} \quad (4.24)$$

- (iii) Orthogonality property of the Extended Chebyshev polynomials of the fourth kind of order k :

$$\langle Z_{(4,n)}^k, Z_{(4,m)}^k \rangle_{w(4,k)} = \begin{cases} 0, & m \neq n, \\ \frac{\pi}{k}, & m = n. \end{cases} \quad (4.25)$$

- (iv) For each $t \in (-1, 1)$

$$\int_{-1}^1 w_{(3,n)}(x) \frac{Z_{(3,n)}^k(x)}{x^k - t^k} dx = \frac{\pi}{k} Z_{(4,n)}^k(t), \quad x \in [-1, 1], \quad (4.26)$$

where $Z_{(3,n)}^k(t)$ is the Extended Chebyshev polynomials of the third kind, $n = 0, 1, 2, \dots$

- (v) For each $t \in (-1, 1)$

$$\int_{-1}^1 w_{(4,n)}(x) \frac{Z_{(4,n)}^k(x)}{x^k - t^k} dx = -\frac{\pi}{k} Z_{(3,n)}^k(t), \quad x \in [-1, 1], \quad (4.27)$$

where $Z_{(4,n)}^k(t)$ is the Extended Chebyshev polynomials of the fourth kind, $n = 0, 1, 2, \dots$

Proof of Lemma 4.3.2:

(i) Using the trigonometric identities

$$\cos(n + \frac{1}{2})\theta + \cos(n - 2 + \frac{1}{2})\theta = 2 \cos \theta \cos(n - 1 + \frac{1}{2})\theta$$

$$\sin(n + \frac{1}{2})\theta + \sin(n - 2 + \frac{1}{2})\theta = 2 \cos \theta \cos(n - 1 + \frac{1}{2})\theta$$

. Dividing both side by $\cos(\frac{1}{2}\theta)$, $\sin(\frac{1}{2}\theta)$ respectively and applying the definition of ECPs of the third and fourth kinds, we get the results.

(ii) Using the relations $Z_{(3,n)}(x) = \frac{\cos((n+\frac{1}{2})\cos^{-1}x^k)}{\cos\frac{1}{2}\cos^{-1}x^k}$ and $Z_{(3,m)}(x) = \frac{\cos((m+\frac{1}{2})\cos^{-1}x^k)}{\cos\frac{1}{2}\cos^{-1}x^k}$, for $n, m = \{0, 1, 2, 3, \dots\}$ and letting $u = x^k \Rightarrow du = kx^{k-1}dx$, we obtain:

$$\begin{aligned} \langle Z_{(3,n)}, Z_{(3,m)} \rangle &= \int_{-1}^1 x^{k-1} \frac{\sqrt{1+x^k}}{\sqrt{1-x^k}} Z_{(3,n)}(x) Z_{(3,m)}(x) dx \\ &= -\frac{1}{k} \int_{\pi}^0 \frac{\sqrt{1+\cos\theta}}{\sqrt{1-\cos\theta}} \frac{\cos((n+\frac{1}{2})\theta)}{\cos\frac{1}{2}\theta} \frac{\cos((m+\frac{1}{2})\theta)}{\cos\frac{1}{2}\theta} \sin(\theta) d\theta \\ &= \frac{1}{k} \int_0^{\pi} \sin(\theta) \frac{\sqrt{1+\cos\theta}}{\sqrt{1-\cos\theta}} \frac{\cos((n+\frac{1}{2})\theta)}{\cos\frac{1}{2}\theta} \frac{\cos((m+\frac{1}{2})\theta)}{\cos\frac{1}{2}\theta} d\theta \\ &= \frac{2}{k} \int_0^{\pi} \sin(\theta) \frac{\sqrt{1+\cos\theta}}{\sqrt{1-\cos\theta}} \frac{\cos((n+\frac{1}{2})\theta)}{\sqrt{1+\cos\theta}} \frac{\cos((m+\frac{1}{2})\theta)}{\sqrt{1+\cos\theta}} d\theta \\ &= \frac{2}{k} \int_0^{\pi} \cos((n+m+1)\theta) + \cos((n-m)\theta); n \neq m \\ &= 0; \end{aligned}$$

The norm of $Z_{(3,n)}$ is given by:

$$\begin{aligned} \|Z_{(3,n)}\|^2 &= \langle Z_{(3,n)}, Z_{(3,n)} \rangle = \frac{2}{k} \int_0^{\pi} \cos^2(n + \frac{1}{2})\theta d\theta \\ &= \frac{\pi}{k}, \quad n = 0, 1, 2, 3, \dots \end{aligned} \quad (4.28)$$

(iii) Substituting the definitions of $Z_{(4,m)}^k(x)$ and $Z_{(4,n)}^k(x)$, for $n, m = \{0, 1, 2, 3, \dots\}$ and

letting $u = x^k \Rightarrow du = kx^{k-1}dx$, we get:

$$\begin{aligned}
 \langle Z_{(4,n)}, Z_{(4,m)} \rangle &= \int_{-1}^1 x^{k-1} \frac{\sqrt{1-x^k}}{\sqrt{1+x^k}} Z_{(4,n)}(x) Z_{(4,m)}(x) dx \\
 &= \frac{1}{k} \int_{-1}^1 \frac{\sqrt{1-u}}{\sqrt{1+u}} \frac{\sin((n+\frac{1}{2})\cos^{-1}u)}{\sin\frac{1}{2}\cos^{-1}u} \frac{\sin((m+\frac{1}{2})\cos^{-1}u)}{\sin\frac{1}{2}\cos^{-1}u} du, \quad u = \cos(\theta) \\
 &= -\frac{1}{k} \int_{\pi}^0 \frac{\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}} \frac{\sin((n+\frac{1}{2})\theta)}{\sin\frac{1}{2}\theta} \frac{\sin((m+\frac{1}{2})\theta)}{\sin\frac{1}{2}\theta} \sin(\theta) d\theta \\
 &= \frac{1}{k} \int_0^{\pi} \sin(\theta) \frac{\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}} \frac{\sin((n+\frac{1}{2})\theta)}{\sin\frac{1}{2}\theta} \frac{\sin((m+\frac{1}{2})\theta)}{\sin\frac{1}{2}\theta} d\theta \\
 &= \frac{2}{k} \int_0^{\pi} \sin(\theta) \frac{\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}} \frac{\sin((n+\frac{1}{2})\theta)}{\sqrt{1-\cos\theta}\sqrt{1-\cos\theta}} \frac{\sin((m+\frac{1}{2})\theta)}{\sqrt{1-\cos\theta}\sqrt{1-\cos\theta}} d\theta \\
 &= \frac{2}{k} \int_0^{\pi} \sin((n+\frac{1}{2})\theta) \sin((m+\frac{1}{2})\theta) d\theta \\
 &= 0;
 \end{aligned}$$

The norm of $Z_{(4,n)}$ is given by:

$$\begin{aligned}
 \|Z_{(4,n)}\|^2 &= \langle Z_{(4,n)}, Z_{(4,n)} \rangle = \frac{2}{k} \int_0^{\pi} \sin^2((n+\frac{1}{2})\theta) d\theta \\
 &= \frac{\pi}{k}, \quad n = 0, 1, 2, 3, \dots
 \end{aligned} \tag{4.29}$$

- (iv) Using the definition of $Z_{(3,n)}^k(x)$ and applying Lemma 4.3.1 together with replacing $w_{(3,k)}(x)$ as in Eq. 4.16 and letting $u = x^k \Rightarrow du = kx^{k-1}dx$ with $x = -1 \Rightarrow u = -1$,

and $x = 1 \Rightarrow u = 1$, we obtain

$$\begin{aligned}
\int_{-1}^1 w_{(3,k)}(x) \frac{Z_{(3,n)}^k(x)}{x^k - t^k} dx &= \int_{-1}^1 x^{k-1} \frac{\sqrt{1+x^k}}{\sqrt{1-x^k}} \frac{Z_{(3,n)}^k(x)}{x^k - t^k} dx \\
&= \int_{-1}^1 x^{k-1} \frac{\sqrt{1+x^k}}{\sqrt{1-x^k}} \frac{\cos((n+\frac{1}{2})\cos^{-1}x^k)}{x^k - t^k \cos \frac{1}{2} \cos^{-1}x^k} dx, \quad u = x^k \\
&= \frac{1}{k} \int_{-1}^1 \frac{\sqrt{1+u}}{\sqrt{1-u}} \frac{\cos((n+\frac{1}{2})\cos^{-1}u)}{u - t^k \cos \frac{1}{2} \cos^{-1}u} du, \quad u = \cos \theta, t^k = \cos(\phi) \\
&= -\frac{1}{k} \int_{\pi}^0 \frac{\sqrt{1+\cos(\theta)}}{\sqrt{1-\cos(\theta)}} \frac{1}{\cos(\theta) - \cos(\phi)} \frac{\cos((n+\frac{1}{2})\theta)}{\cos \frac{1}{2} \theta} \sin \theta d\theta \\
&= \frac{1}{k} \int_0^{\pi} \frac{\sqrt{1+\cos(\theta)}}{\sqrt{1-\cos(\theta)}} \frac{1}{\cos(\theta) - \cos(\phi)} \frac{\cos((n+\frac{1}{2})\theta)}{\cos^2 \frac{1}{2} \theta} \cos \frac{1}{2} \theta \sin \theta d\theta \\
&= \frac{2}{k} \int_0^{\pi} \frac{\sqrt{1+\cos(\theta)}}{\sqrt{1-\cos(\theta)}} \frac{1}{\cos(\theta) - \cos(\phi)} \frac{\cos((n+\frac{1}{2})\theta)}{1+\cos \theta} \cos \frac{1}{2} \theta \sin \theta d\theta \\
&= \frac{2}{k} \int_0^{\pi} \frac{\cos(n+\frac{1}{2})\theta}{\cos(\theta) - \cos(\phi)} \cos \frac{1}{2} \theta d\theta, \quad \text{by applying Lemma 4.3.1} \\
&= \frac{2\pi \sin(n+\frac{1}{2})\phi}{k \sin(\frac{1}{2}\phi)} \\
&= \frac{\pi \sin(n+\frac{1}{2})\cos^{-1}t^k}{k \sin(\frac{1}{2}\cos^{-1}t^k)} \\
&= \frac{\pi}{k} Z_{(4,n)}^k(t)
\end{aligned}$$

where $n = 0, 1, 2, 3, \dots, k$ is positive odd integer, when $t \in (-1, 1) \Rightarrow \phi(t) = t^k \in (-1, 1)$ since k is positive odd integer, so our assumption is well defined.

- (v) Substituting $Z_{(4,n)}^k(x)$ and applying Lemma 4.3.1 together with replacing $w_{(4,k)}(x)$ as in Eq. (4.16) and letting $u = x^k \Rightarrow du = kx^{k-1}dx$ with $x = -1 \Rightarrow u = -1$, and

$x = 1 \Rightarrow u = 1$, we have

$$\begin{aligned}
 \int_{-1}^1 w_{(4,k)}(x) \frac{Z_{(4,n)}^k(x)}{x^k - t^k} dx &= \int_{-1}^1 x^{k-1} \frac{\sqrt{1-x^k} Z_{(4,n)}^k(x)}{\sqrt{1+x^k} (x^k - t^k)} dx \\
 &= \int_{-1}^1 x^{k-1} \frac{\sqrt{1-x^k} \sin((n+\frac{1}{2}) \cos^{-1} x^k)}{\sqrt{1+x^k} (x^k - t^k \sin^{\frac{1}{2}} \cos^{-1} x^k)} dx, \quad u = x^k \\
 &= \frac{1}{k} \int_{-1}^1 \frac{\sqrt{1-u} \sin((n+\frac{1}{2}) \cos^{-1} u)}{\sqrt{1+u} (u - t^k \sin^{\frac{1}{2}} \cos^{-1} u)} dx, \quad u = \cos \theta, t^k = \cos(\phi) \\
 &= -\frac{1}{k} \int_{\pi}^0 \frac{\sqrt{1-\cos(\theta)}}{\sqrt{1+\cos(\theta)}} \frac{1}{\cos(\theta) - \cos(\phi)} \frac{\sin((n+\frac{1}{2})\theta)}{\sin^{\frac{1}{2}} \theta} \sin \theta d\theta \\
 &= \frac{1}{k} \int_0^{\pi} \frac{\sqrt{1-\cos(\theta)}}{\sqrt{1+\cos(\theta)}} \frac{1}{\sin(\theta) - \cos(\phi)} \frac{\sin((n+\frac{1}{2})\theta)}{\sin^2 \frac{1}{2} \theta} \sin \frac{1}{2} \theta \sin \theta d\theta \\
 &= \frac{2}{k} \int_0^{\pi} \frac{\sqrt{1-\cos(\theta)}}{\sqrt{1+\cos(\theta)}} \frac{1}{\cos(\theta) - \cos(\phi)} \frac{\sin((n+\frac{1}{2})\theta)}{1 - \cos \theta} \sin \frac{1}{2} \theta \sin \theta d\theta \\
 &= \frac{2}{k} \int_0^{\pi} \frac{\sin(n+\frac{1}{2})\theta}{\cos(\theta) - \cos(\phi)} \sin \frac{1}{2} \theta d\theta, \quad \text{applying Lemma 4.3.1} \\
 &= -\frac{2\pi \cos(n+\frac{1}{2})\phi}{k \sin(\frac{1}{2}\phi)} \\
 &= -\frac{\pi \cos(n+\frac{1}{2}) \cos^{-1} t^k}{k \sin(\frac{1}{2} \cos^{-1} t^k)} \\
 &= -\frac{\pi}{k} Z_{(3,n)}^k(t)
 \end{aligned}$$

where $n = 1, 2, 3, \dots$; k is positive odd integer.

4.4 Relations Between Extended Chebyshev Polynomials of the First - Fourth Kinds of Order k

Recurrence relations and relations are fundamentally important in many branches of mathematics. In this section, we will discuss the connections between each other's Extended Chebyshev polynomials, and how they are related in recursive techniques. These relations can be derived into sequences based on identities of trigonometric functions.

The recursive relationship in Lemmas 3.3.3 and 4.4.1 can be used to compute the value of the Extended Chebyshev polynomials for all $n = 0, 1, 2, 3, \dots$; $x \in [-1, 1]$ by knowing only the first two terms of the Extended Chebyshev polynomials. From the discussion in chapters (3) and (4), we found that the properties of $(Z_{(i,n)}^k(x))$, $i = \{1, 2, 3, 4\}$. The main relations and recurrence relations can be summarized in the following Lemma:

Lemma 4.4.1 Let $Z_{(i,n)}^k(x), i = \{1, 2, 3, 4\}$ be the set of the Extended Chebyshev polynomials of the first - fourth kinds of order k , respectively, the following relations are satisfied:

- (i) The relations between the Extended Chebyshev polynomials of the second, third and fourth kinds of order k are:

$$Z_{(2,n)}^k(x) = \frac{1}{2} [Z_{(3,n)}^k(x) + Z_{(4,n)}^k(x)]. \quad (4.30)$$

- (ii) The three terms of recurrence relations between the Extended Chebyshev polynomials of the second and third kinds are:

$$Z_{(3,n)}^k(x) = Z_{(2,n)}^k(x) - Z_{(2,n-1)}^k(x). \quad (4.31)$$

- (iii) The three terms of recurrence relations between the Extended Chebyshev polynomials of the second and fourth kinds are:

$$Z_{(4,n)}^k(x) = Z_{(2,n)}^k(x) + Z_{(2,n-1)}^k(x). \quad (4.32)$$

- (iv) The three terms of recurrence relations between the Extended Chebyshev polynomials of the first, third and fourth kinds are

$$2Z_{(1,n)}^k(x) = Z_{(4,n)}^k(x) - Z_{(4,n-1)}^k(x) = Z_{(3,n)}^k(x) + Z_{(3,n-1)}^k(x). \quad (4.33)$$

Proof Lemma 4.4.1

- (i) The result follows directly from the trigonometric definitions

$$\begin{aligned} Z_{(2,n)}^k(x) &= \frac{\sin(n+1)\theta}{\sin \theta} = \frac{\sin(n + \frac{1}{2} + \frac{1}{2})\theta}{\sin \theta} \\ &= \frac{\sin(n + \frac{1}{2})\theta \cos \frac{1}{2}\theta + \cos(n + \frac{1}{2})\theta \sin \frac{1}{2}\theta}{\sin 2\frac{1}{2}\theta} \\ &= \frac{\sin(n + \frac{1}{2})\theta \cos \frac{1}{2}\theta + \cos(n + \frac{1}{2})\theta \sin \frac{1}{2}\theta}{2 \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta} \\ &= \frac{1}{2} \left[\frac{\cos(n + \frac{1}{2})\theta}{\cos \frac{1}{2}\theta} + \frac{\sin(n + \frac{1}{2})\theta}{\sin \frac{1}{2}\theta} \right] \end{aligned} \quad (4.34)$$

where $n = 0, 1, 2, 3, \dots$

(ii) We can start from the trigonometric identity:

$$\begin{aligned}
 2 \sin \frac{1}{2} \theta \cos(n + \frac{1}{2}) \theta &= \sin(n + 1) \theta - \sin n \theta, \\
 2 \frac{\sin \frac{1}{2} \theta \cos(n + \frac{1}{2}) \theta}{\sin \theta} &= \frac{\sin(n + 1) \theta}{\sin \theta} - \frac{\sin n \theta}{\sin \theta} \quad \text{divided both side by } \sin \theta \\
 2 \frac{\sin \frac{1}{2} \theta \cos(n + \frac{1}{2}) \theta}{\sin 2 \frac{1}{2} \theta} &= \frac{\sin(n + 1) \theta}{\sin \theta} - \frac{\sin n \theta}{\sin \theta} \\
 2 \frac{\sin \frac{1}{2} \theta \cos(n + \frac{1}{2}) \theta}{2 \sin \frac{1}{2} \theta \cos \frac{1}{2} \theta} &= \frac{\sin(n + 1) \theta}{\sin \theta} - \frac{\sin n \theta}{\sin \theta} \\
 \frac{\cos(n + \frac{1}{2}) \theta}{\cos \frac{1}{2} \theta} &= \frac{\sin(n + 1) \theta}{\sin \theta} - \frac{\sin(n + 1 - 1) \theta}{\sin \theta},
 \end{aligned} \tag{4.35}$$

Using the definitions of the Extended Chebyshev polynomials of the second and third kinds, the proof is completed.

(iii) Using the trigonometric identity:

$$\begin{aligned}
 2 \cos \frac{1}{2} \theta \sin(n + \frac{1}{2}) \theta &= \sin(n + 1) \theta - \sin -n \theta, \\
 2 \frac{\cos \frac{1}{2} \theta \sin(n + \frac{1}{2}) \theta}{\sin \theta} &= \frac{\sin(n + 1) \theta}{\sin \theta} + \frac{\sin n \theta}{\sin \theta} \quad \text{divided both side by } \sin \theta \\
 2 \frac{\cos \frac{1}{2} \theta \sin(n + \frac{1}{2}) \theta}{\sin 2 \frac{1}{2} \theta} &= \frac{\sin(n + 1) \theta}{\sin \theta} + \frac{\sin n \theta}{\sin \theta} \\
 2 \frac{\cos \frac{1}{2} \theta \sin(n + \frac{1}{2}) \theta}{2 \sin \frac{1}{2} \theta \cos \frac{1}{2} \theta} &= \frac{\sin(n + 1) \theta}{\sin \theta} + \frac{\sin n \theta}{\sin \theta} \\
 \frac{\sin(n + \frac{1}{2}) \theta}{\sin \frac{1}{2} \theta} &= \frac{\sin(n + 1) \theta}{\sin \theta} + \frac{\sin(n + 1 - 1) \theta}{\sin \theta}
 \end{aligned} \tag{4.36}$$

Using the definitions of the Extended Chebyshev polynomials of the second and forth kinds, the proof is completed.

(iv) Using the definition of the Extended Chebyshev polynomials of the fourth kind, we find:

$$\begin{aligned}
Z_{(4,n)}^k(x) - Z_{(4,n-1)}^k(x) &= \frac{\sin(n + \frac{1}{2})\theta}{\sin(\frac{1}{2}\theta)} - \frac{\sin(n - 1 + \frac{1}{2})\theta}{\sin(\frac{1}{2}\theta)} \\
&= \frac{\sin(n + \frac{1}{2})\theta - \sin(n - 1 + \frac{1}{2})\theta}{\sin(\frac{1}{2}\theta)} \\
&= 2 \frac{\cos(\frac{(n + \frac{1}{2} + n - \frac{1}{2})}{2})\theta \sin(\frac{(n + \frac{1}{2} - (n - \frac{1}{2}))}{2})\theta}{\sin(\frac{1}{2}\theta)} \\
&= 2 \frac{\cos(n)\theta \sin(\frac{1}{2}\theta)}{\sin(\frac{1}{2}\theta)} \\
&= 2Z_{(1,n)}^k(x)
\end{aligned} \tag{4.37}$$

Moreover:

$$\begin{aligned}
Z_{(3,n)}^k(x) + Z_{(3,n-1)}^k(x) &= \frac{\cos(n + \frac{1}{2})\theta}{\cos(\frac{1}{2}\theta)} + \frac{\cos(n - 1 + \frac{1}{2})\theta}{\cos(\frac{1}{2}\theta)} \\
&= \frac{\cos(n + \frac{1}{2})\theta + \cos(n - 1 + \frac{1}{2})\theta}{\cos(\frac{1}{2}\theta)} \\
&= 2 \frac{\cos(\frac{(n + \frac{1}{2} + n - \frac{1}{2})}{2})\theta \cos(\frac{(n + \frac{1}{2} - (n - \frac{1}{2}))}{2})\theta}{\cos(\frac{1}{2}\theta)} \\
&= 2 \frac{\cos(n)\theta \cos(\frac{1}{2}\theta)}{\cos(\frac{1}{2}\theta)} \\
&= 2Z_{(1,n)}^k(x)
\end{aligned} \tag{4.38}$$

We immediately complete the proof of Lemma 4.3.2.

4.5 Function Approximation Via Extended Chebyshev Polynomials of The Third and Fourth Kinds of Order k

In this section, we will continue our study of special singular integral equation of order k (SSIEk) with two other cases, (case (III) and case (IV)).

Let us consider the linear SSIEk of first kind in the form:

$$\int_{-1}^1 \frac{f(x)}{x^k - t^k} dx = g(t^k), t \in (-1, 1) \tag{4.39}$$

where k is positive odd integer. In particular, when $k = 1$ Eq. (4.39) reduces to the standard airfoil equation in aerodynamics and it has four types of analytical solutions (for more

detail see Eshkuvatov et al. (2009)). The main idea is dealing with singular kernels and reducing the IE problem into algebraic equations that involve matrices. The key point is to approximate the unknown function as finite series of the Extended Chebyshev polynomials of the third and fourth kinds of order k , $Z_{(i,n)}^k(t)$, $i = \{3, 4\}$, $n = 0, 1, 2, \dots$.

For convenience, let us rewrite Eq. (4.39) as:

$$\int_{-1}^1 M(x, t) f(x) dx = g(t^k), \quad t \in (-1, 1), \quad (4.40)$$

where the right hand side $g(t^k)$ is known, and $M(x, t)$ is given by:

$$M(x, t) = \frac{1}{x^k - t^k}, \quad k \text{ is positive odd integers} \quad (4.41)$$

and $f(x)$ needs to be determined. The method for solving Eq. (4.40) is based on expansions of the approximate solution into Extended Chebyshev polynomials of the third and fourth kinds $Z_{(i,n)}^k(x)$, $i = \{3, 4\}$ with three and four cases:

Case (III): The solution is bounded at the point $x = -1$.

For the semi bounded solution of Eq.(4.40), we approximate $f(t)$ by:

$$f_n(t) = a_0 w_{(3,k)}(t) Z_{(3,0)}^k(t) + w_{(3,k)}(t) \sum_{j=1}^n a_j Z_{(3,j)}^k(t) \quad (4.42)$$

where k is positive odd integers and $w_{(3,k)}(t)$ as defined in Eq. 4.16.

Substitute Eq. (4.42) into Eq. (4.40)

$$\begin{aligned} \int_{-1}^1 M(x, t) f(x) dx &= g(t^k) \\ a_0 \int_{-1}^1 \frac{1}{x^k - t^k} w_{(3,k)}(x) Z_{(3,0)}^k(x) dx + \int_{-1}^1 \frac{1}{x^k - t^k} w_{(3,k)}(x) \sum_{j=1}^n a_j Z_{(3,j)}^k(x) dx &= g(t^k) \end{aligned} \quad (4.43)$$

$$a_0 \int_{-1}^1 \frac{1}{x^k - t^k} w_{(3,k)}(x) Z_{(3,0)}^k(x) dx + \sum_{j=1}^n \int_{-1}^1 a_j \frac{1}{x^k - t^k} w_{(3,k)}(x) Z_{(3,j)}^k(x) dx = g(t^k)$$

$$\begin{aligned} a_0 \int_{-1}^1 \frac{w_{(3,k)}(x)}{x^k - t^k} Z_{(3,0)}^k(x) dx + a_1 \int_{-1}^1 \frac{w_{(3,k)}(x)}{x^k - t^k} Z_{(3,1)}^k(x) dx + a_2 \int_{-1}^1 \frac{w_{(3,k)}(x)}{x^k - t^k} Z_{(3,2)}^k(x) dx + \dots \\ + a_n \int_{-1}^1 \frac{w_{(3,k)}(x)}{x^k - t^k} Z_{(3,n)}^k(x) dx = g(t^k) \end{aligned} \quad (4.44)$$

Use Lemma 4.3.2 (vii) Eq. 4.44 reduce to:

$$a_0 \frac{\pi}{k} Z_{(4,0)}^k(t) + a_1 \frac{\pi}{k} Z_{(4,1)}^k(t) + a_2 \frac{\pi}{k} Z_{(4,2)}^k(t) + \cdots + a_n \frac{\pi}{k} Z_{(4,n)}^k(t) = g(t^k) \quad (4.45)$$

We use the classical Galerkin method to get $(n+1)$ unknowns and the same number of equations, we multiply both sides of Eq. (4.45) by $Z_{(4,m)}^k(t), m = 0, 1, 2, \dots, n$, under the weight function $w_{(4,k)}(t)$ then integrating over the interval $[-1, 1]$, Eq. (4.45) becomes:

$$\begin{aligned} \frac{\pi}{k} a_0 \int_{-1}^1 w_{(4,k)}(x) Z_{(4,0)}^k(t) Z_{(4,m)}^k(t) dt + \frac{\pi}{k} a_1 \int_{-1}^1 w_{(4,k)}(x) Z_{(4,1)}^k(t) Z_{(4,m)}^k(t) dt + \cdots \\ + \frac{\pi}{k} a_n \int_{-1}^1 w_{(4,k)}(x) Z_{(4,n)}^k(t) Z_{(4,m)}^k(t) dt \quad (4.46) \\ = \int_{-1}^1 w_{(4,k)}(x) g(t^k) Z_{(4,m)}^k(t) dt \end{aligned}$$

for $m = 0, 1, 2, 3, \dots, n$.

For $m = 0$

$$\begin{aligned} \frac{\pi}{k} a_0 \int_{-1}^1 w_{(4,k)}(x) Z_{(4,0)}^k(t) Z_{(4,0)}^k(t) dt + \frac{\pi}{k} a_1 \int_{-1}^1 w_{(4,k)}(x) Z_{(4,1)}^k(t) Z_{(4,0)}^k(t) dt + \cdots \\ + \frac{\pi}{k} a_n \int_{-1}^1 w_{(4,k)}(x) Z_{(4,n)}^k(t) Z_{(4,0)}^k(t) dt \quad (4.47) \\ = \int_{-1}^1 w_{(4,k)}(x) g(t^k) Z_{(4,0)}^k(t) dt \end{aligned}$$

Applying of the orthogonality property of the Extended Chebyshev polynomials of the fourth kind $Z_{(4,m)}^k(t), m = 0, 1, 2, 3, \dots, n$, Lemma 4.3.2 (vii), Eq. (4.47) yields:

$$\begin{aligned} \frac{\pi}{k} \frac{\pi}{k} a_0 + 0 + \cdots + 0 &= \int_{-1}^1 w_{(4,k)}(x) g(t^k) Z_{(4,0)}^k(t) dt \\ \frac{\pi^2}{k^2} a_0 &= \int_{-1}^1 w_{(4,k)}(x) g(t^k) Z_{(4,0)}^k(t) dt \end{aligned}$$

For $m = 1$

$$\begin{aligned} \frac{\pi}{k} a_0 \int_{-1}^1 w_{(4,k)}(x) Z_{(4,0)}^k(t) Z_{(4,1)}^k(t) dt + \frac{\pi}{k} a_1 \int_{-1}^1 w_{(4,k)}(x) Z_{(4,1)}^k(t) Z_{(4,1)}^k(t) dt + \cdots \\ + \frac{\pi}{k} a_n \int_{-1}^1 w_{(4,k)}(x) Z_{(4,n)}^k(t) Z_{(4,1)}^k(t) dt \quad (4.48) \\ = \int_{-1}^1 w_{(4,k)}(x) g(t^k) Z_{(4,1)}^k(t) dt \end{aligned}$$

Using orthogonality property Lemma 4.3.2 (vii), Eq. (4.48) becomes:

$$0 + \frac{\pi}{k} \frac{\pi}{k} a_1 + 0 + \dots + 0 = \int_{-1}^1 w_{(4,k)}(x) g(t^k) Z_{(4,1)}^k(t) dt$$

$$\frac{\pi^2}{k^2} a_1 = \int_{-1}^1 w_{(4,k)}(x) g(t^k) Z_{(4,1)}^k(t) dt$$

Proceeding in the same way, we have:

For $m = n$

$$\begin{aligned} \frac{\pi}{k} a_0 \int_{-1}^1 w_{(4,k)}(x) Z_{(4,n)}^k(t) Z_{(4,1)}^k(t) dt + \frac{\pi}{k} a_1 \int_{-1}^1 w_{(4,k)}(x) Z_{(4,1)}^k(t) Z_{(4,n)}^k(t) dt + \dots \\ + \frac{\pi}{k} a_n \int_{-1}^1 w_{(4,k)}(x) Z_{(4,n)}^k(t) Z_{(4,n)}^k(t) dt \quad (4.49) \\ = \int_{-1}^1 w_{(4,k)}(x) g(t^k) Z_{(4,n)}^k(t) dt \end{aligned}$$

Applying orthogonality property Lemma 4.3.2 (v), Eq. (4.49) becomes:

$$0 + 0 + \dots + 0 + \frac{\pi}{k} \frac{\pi}{k} a_n = \int_{-1}^1 w_{(4,k)}(x) g(t^k) Z_{(4,n)}^k(t) dt$$

$$\frac{\pi^2}{k^2} a_n = \int_{-1}^1 w_{(4,k)}(x) g(t^k) Z_{(4,n)}^k(t) dt$$

The linear simultaneous equations generated from Eq. (4.47) - (4.49) formulate a system of linear algebraic equations with $(n+1)$ unknown coefficients:

$$a_m = \frac{k^2}{\pi^2} \langle g, Z_{(1,m)}^k \rangle_{w_{(4,k)}} \quad m = 0, 1, 2, 3, \dots, n. \quad (4.50)$$

Eq. 4.50 formulates a linear of system equations which can be represented as matrix form:

$$\begin{bmatrix} \frac{\pi^2}{k^2} & 0 & 0 & 0 & \dots & 0 \\ 0 & \frac{\pi^2}{k^2} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\pi^2}{k^2} & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \frac{\pi^2}{k^2} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_m \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} C_0^k \\ C_1^k \\ \vdots \\ C_m^k \\ \vdots \\ C_n^k \end{bmatrix} \quad (4.51)$$

where $m = 0, 1, 2, \dots, n$, and

$$C_m^k = \langle g, Z_{(4,m)}^k \rangle_{w_{(4,k)}(x)} \quad (4.52)$$

Case (IV): The solution is bounded at the point $x = 1$.

For the semi bounded solution of Eq.(4.40) proceed in the same way (Case (III)) and approximate $f(t)$ by:

$$f_n(t) = a_0 w_{(4,k)}(t) Z_{(4,0)}^k(t) + w_{(4,k)}(t) \sum_{j=1}^n a_j Z_{(4,j)}^k(t) \quad (4.53)$$

where k is positive odd integers and $w_{(4,k)}(t)$ defined as in Eq. 4.16. Substitute Eq. (4.53) into Eq. (4.40)

$$\begin{aligned} \int_{-1}^1 M(x, t) f(x) dx &= g(t^k) \\ a_0 \int_{-1}^1 \frac{1}{x^k - t^k} w_{(4,k)}(x) Z_{(4,0)}^k(x) dx + \int_{-1}^1 \frac{1}{x^k - t^k} w_{(4,k)}(x) \sum_{j=1}^n a_j Z_{(4,j)}^k(x) dx &= g(t^k) \\ a_0 \int_{-1}^1 \frac{1}{x^k - t^k} w_{(4,k)}(x) Z_{(4,0)}^k(x) dx + \sum_{j=1}^n \int_{-1}^1 \frac{1}{x^k - t^k} w_{(4,k)}(x) a_j Z_{(4,j)}^k(x) dx &= g(t^k) \\ a_0 \int_{-1}^1 \frac{w_{(4,k)}(x)}{x^k - t^k} Z_{(4,0)}^k(x) dx + a_1 \int_{-1}^1 \frac{w_{(4,k)}(x)}{x^k - t^k} Z_{(4,1)}^k(x) dx + a_2 \int_{-1}^1 \frac{w_{(4,k)}(x)}{x^k - t^k} Z_{(4,2)}^k(x) dx + \dots \\ + a_n \int_{-1}^1 \frac{w_{(4,k)}(x)}{x^k - t^k} Z_{(4,n)}^k(x) dx &= g(t^k) \end{aligned} \quad (4.54)$$

$$(4.55)$$

Use Lemma 4.3.2 (viii) Eq. 4.55 reduces to:

$$-a_0 \frac{\pi}{k} Z_{(3,0)}^k(t) - a_1 \frac{\pi}{k} Z_{(3,1)}^k(t) - a_2 \frac{\pi}{k} Z_{(3,2)}^k(t) - \dots - a_n \frac{\pi}{k} Z_{(3,n)}^k(t) = g(t^k) \quad (4.56)$$

To determine the unknown coefficients $a_m, m = 0, 1, 2, 3, \dots, n$, we use the classical Galerkin method. We multiply both sides of Eq. (4.56) by Extended Chebyshev polynomials

$Z_{(3,m)}^k(t), m = 0, 1, 2, \dots, n$, under the weight function $w_{(3,k)}(t)$ then integrating over the interval $[-1, 1]$, Eq. (4.56) becomes:

$$\begin{aligned} & -\frac{\pi}{k}a_0 \int_{-1}^1 w_{(3,k)}(x)Z_{(3,0)}^k(t)Z_{(3,m)}^k(t)dt - \frac{\pi}{k}a_1 \int_{-1}^1 w_{(3,k)}(x)Z_{(3,1)}^k(t)Z_{(3,m)}^k(t)dt - \dots \\ & -\frac{\pi}{k}a_n \int_{-1}^1 w_{(3,k)}(x)Z_{(3,n)}^k(t)Z_{(3,m)}^k(t)dt \\ & = \int_{-1}^1 w_{(3,k)}(x)g(t^k)Z_{(3,m)}^k(t)dt \end{aligned} \quad (4.57)$$

for $m = 0, 1, 2, 3, \dots, n$.

For $m = 0$

$$\begin{aligned} & -\frac{\pi}{k}a_0 \int_{-1}^1 w_{(3,k)}(x)Z_{(3,0)}^k(t)Z_{(3,0)}^k(t)dt - \frac{\pi}{k}a_1 \int_{-1}^1 w_{(3,k)}(x)Z_{(3,1)}^k(t)Z_{(3,0)}^k(t)dt - \dots \\ & -\frac{\pi}{k}a_n \int_{-1}^1 w_{(3,k)}(x)Z_{(3,n)}^k(t)Z_{(3,0)}^k(t)dt \\ & = \int_{-1}^1 w_{(3,k)}(x)g(t^k)Z_{(3,0)}^k(t)dt \end{aligned} \quad (4.58)$$

Applying the orthogonality property of Extended Chebyshev polynomials of the fourth kind $Z_{(3,m)}^k(t), m = 0, 1, 2, 3, \dots, n$. Lemma 4.3.2 (vi), Eq. (4.58) yields:

$$\begin{aligned} & -\frac{\pi}{k} \frac{\pi}{k} a_0 - 0 - \dots - 0 = \int_{-1}^1 w_{(3,k)}(x)g(t^k)Z_{(3,0)}^k(t)dt \\ & \frac{\pi^2}{k^2} a_0 = \int_{-1}^1 w_{(3,k)}(x)g(t^k)Z_{(3,0)}^k(t)dt \end{aligned}$$

For $m = 1$

$$\begin{aligned} & -\frac{\pi}{k}a_0 \int_{-1}^1 w_{(3,k)}(x)Z_{(3,0)}^k(t)Z_{(3,1)}^k(t)dt - \frac{\pi}{k}a_1 \int_{-1}^1 w_{(3,k)}(x)Z_{(3,1)}^k(t)Z_{(3,1)}^k(t)dt - \dots \\ & -\frac{\pi}{k}a_n \int_{-1}^1 w_{(3,k)}(x)Z_{(3,n)}^k(t)Z_{(3,1)}^k(t)dt \\ & = \int_{-1}^1 w_{(3,k)}(x)g(t^k)Z_{(3,1)}^k(t)dt \end{aligned} \quad (4.59)$$

Using of the orthogonality property (Lemma 4.3.2 (vi)), Eq. (4.59) becomes:

$$-0 - \frac{\pi}{k} \frac{\pi}{k} a_1 - 0 - \dots - 0 = \int_{-1}^1 w_{(3,k)}(x) g(t^k) Z_{(3,1)}^k(t) dt$$

$$\frac{\pi^2}{k^2} a_1 = \int_{-1}^1 w_{(3,k)}(x) g(t^k) Z_{(3,1)}^k(t) dt$$

Proceeding in the same way, we have

For $m = n$

$$-\frac{\pi}{k} a_0 \int_{-1}^1 w_{(3,k)}(x) Z_{(3,n)}^k(t) Z_{(3,1)}^k(t) dt - \frac{\pi}{k} a_1 \int_{-1}^1 w_{(3,k)}(x) Z_{(3,1)}^k(t) Z_{(3,n)}^k(t) dt - \dots$$

$$-\frac{\pi}{k} a_n \int_{-1}^1 w_{(3,k)}(x) Z_{(3,n)}^k(t) Z_{(3,n)}^k(t) dt$$

$$= \int_{-1}^1 w_{(3,k)}(x) g(t^k) Z_{(3,n)}^k(t) dt \quad (4.60)$$

Applying of the orthogonality property (Lemma 4.3.2 (vi)), Eq. (4.60) becomes:

$$-0 - 0 - \dots - 0 - \frac{\pi}{k} \frac{\pi}{k} a_n = \int_{-1}^1 w_{(3,k)}(x) g(t^k) Z_{(3,n)}^k(t) dt$$

$$\frac{\pi^2}{k^2} a_n = - \int_{-1}^1 w_{(3,k)}(x) g(t^k) Z_{(3,n)}^k(t) dt$$

The linear simultaneous equations generated from Eq. (4.56) - (4.60) formulate a system of linear algebraic equations with $(n+1)$ unknown coefficients:

$$a_m = -\frac{k^2}{\pi^2} \langle g, Z_{(3,m)}^k \rangle_{w_{(3,k)}} \quad m = 0, 1, 2, 3, \dots, n \quad (4.61)$$

We use MATLAB to determine the coefficients $a_m, m = 0, 1, 2, 3, \dots, n$.

4.6 Conclusion

Two families of the Extended Chebyshev polynomials of the third and fourth kinds of order $k, Z_{(i,n)}^k(x), i = \{3, 4\}$ have been developed. It showed some graphs of the ECPs of the of third and fourth kinds of order k . The zeros of ECPs of the third and fourth kinds of order k are discussed and described.

Recurrence relations and the relations between the other two kinds are proved. We state some of the associated relations of the two kinds. The behavior of Extended Chebyshev sequences of the third and fourth kinds are described when k is sufficient large odd integers. It is found that Extended Chebyshev sequences of the third and fourth kinds converge approximately into two values $\{-1, 1\}$ depending on index n .

The ECPs of the third and fourth kinds $Z_{(i,n)}^k(x), i = \{3, 4\}$ are orthogonal with respect to weight functions, $w_{(3,k)}(x) = x^{k-1} \sqrt{\frac{1+x^k}{1-x^k}}$ and $w_{(4,k)}(x) = x^{k-1} \sqrt{\frac{1-x^k}{1+x^k}}$ respectively, on the interval $[-1, 1]$, where k is positive odd integers. Spectral properties of the ECPs of the third and fourth kinds $Z_{(i,n)}^k(x), i = \{3, 4\}$ are proved. We also demonstrate how to use these orthogonal polynomials to aid in the solution of SSIEk. The unknown functions are characterized in terms of truncated series of ECPs. Effective numerical methods are proposed to evaluate the density functions. Furthermore, a study of effective and accurate approximate methods for solving the SSIEk of the third and fourth kinds with third and fourth cases over a finite interval $[-1, 1]$ is proposed.

UNIVERSITI SAINS ISLAM MALAYSIA
 جامعة العلوم الإسلامية
 ISLAMIC SCIENCE UNIVERSITY OF MALAYSIA