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# Solving Duffing Type Differential Equations using a Three-Point Block Variable Order Step Size Method

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**Abstract.** This research proposes a three-point block method for solving Duffing type higher order ordinary differential equations (ODEs) which is also commonly referred as the Duffing oscillator. The research conducted implements a variable order step size technique for approximating the exact solution for the Duffing Oscillator. The proposed algorithm will be tested against various Duffing oscillators and numerical approximation will be compared with current viable methods. The accuracy and efficiency of the proposed method will be illustrated in the numerical results.

## 1. Introduction

Non-linear initial value Duffing oscillator with an external force has many applications in physics and engineering, which includes the detection weak signals, such as the characteristic signal of early machinery fault. The Duffing oscillator is governed by the higher order ordinary differential equation (ODE),

$$y'' + \delta y' + \alpha y + \beta y^3 = \gamma \cos(\omega t)$$
$$y(0) = a, \quad y'(0) = b,$$

with  $\delta$ ,  $\alpha$  and  $\beta$  as coefficients,  $\gamma$  and  $\omega$  respectively the amplitude and frequency of the external force. Previously, the popular approach for solving these higher order ODEs was by reducing them to a system of first order ODEs and solved using first order methods. Due to their efficiency, methods for approximating higher order ODEs were disregarded as robust. Works by [1–3] were able to revive interest of researchers in solving higher order ODEs by introducing theoretically development of direct methods. Research related to numerical solution of higher order ODEs directly using multistep method can be obtained in the following articles [4–12]. The research conducted in this article was inspired by [4]. A variable order step size multistep method was developed in [4] using a divided difference formulation known as Direct Integration (DI) method. Although the DI method was shown to be efficient, the need to calculate the integration coefficients at every step change hindered its full potential.



The need for calculating integration coefficients at every step change was overcome in [8] by using a backward difference formulation. The backward difference formulation [8] was also fitted with a variable order step size algorithm similar to [4]. Mohd Ijam [13] then extended the works of [7] by implementing a two-point block formulation. In the current research, we extend the works of [8] and Mohd Ijam [13] by implementing a variable order step size three-point block method using a backward difference formulation in predictor-corrector form. The following section details the derivation of the integration coefficients for the proposed method.

## 2. Explicit and Implicit Integration Coefficients

Derivation of the one-point and two-point block integration coefficients can be found in [8] and [13]. In formulating the three-point block integration coefficients, we begin with the derivation for the predictor (explicit). Now, consider a second order ordinary differential equation

$$y'' = f(t, y, y') \quad (1)$$

with the initial value conditions

$$y(a) = \eta, \quad y'(a) = \eta',$$

in the interval of  $a \leq t \leq b$ . Integrating the second order ODE once yields

$$y'(t_{n+3}) = y'(t_n) + \int_{t_n}^{t_{n+3}} f(t, y, y') dt.$$

By interpolating the  $k$  values  $(t_n, f_n), (t_{n-1}, f_{n-1}), \dots, (t_{n-k}, f_{n-k})$ , using the Gregory-Newton polynomial,

$$P_n(t) = \sum_{i=0}^{k-1} \binom{-s}{i} \nabla^i f_n,$$

gives the following estimate

$$y'(t_{n+3}) = y'(t_n) + \int_0^3 \sum_{i=0}^{k-1} \binom{-s}{i} \nabla^i f_n ds,$$

where

$$t = t_n + sh.$$

Let  $G_{3,1}(t)$  be a function that generates the set of coefficients  $\sigma_{3,1,i}$  and is defined as

$$G_{3,1}(t) = \sum_{i=0}^{\infty} \sigma_{3,1,i} t^i, \quad (2)$$

with the coefficients

$$\sigma_{3,1,i} = (-1)^i \int_0^3 \binom{-s}{i} ds. \quad (3)$$

By substituting (3) into (2) then solving the integral, the generating function can be written in the following form

$$G_{3,1}(t) = - \left[ \frac{(1-t)^{-3}}{\log(1-t)} - \frac{1}{\log(1-t)} \right]. \quad (4)$$

Next, integrating (1) twice yields,

$$y(t_{n+3}) = y(t_n) + (3h)y'(t_n) + \int_{t_n}^{t_{n+3}} (t_{n+3} - t) f(t, y, y') dt.$$

To obtain the second order generating function,  $G_{3,2}(t)$  the mathematical process above is repeated using the set of coefficients,  $\sigma_{3,2,i}$ . The following is the formulation obtained for  $G_{3,2}(t)$ ,

$$G_{3,2}(t) = \left[ \frac{3}{\log(1-t)} - \frac{G_{3,1}(t)}{\log(1-t)} \right].$$

The second half of the derivation is the integration coefficients for the corrector (implicit). The implicit coefficients can be attained in a similar manner as the explicit coefficients but with the slight difference of choosing

$$t = t_{n+3} + sh.$$

This changes the limit of integration for the implicit coefficients,

$$\sigma_{3,1,i}^* = (-1)^i \int_{-3}^0 \binom{-s}{i} ds,$$

hence, producing the first order implicit generating function

$$G_{3,1}^*(t) = - \left[ \frac{1}{\log(1-t)} - \frac{(1-t)^3}{\log(1-t)} \right].$$

The second order generating function is then derived in the form of

$$G_{3,2}^*(t) = \left[ \frac{3(1-t)^3}{\log(1-t)} - \frac{G_{3,1}^*(t)}{\log(1-t)} \right].$$

Through mathematical induction, the following relationship between explicit and implicit generating function is established

$$G_{3,i}^*(t) = (1-t)^3 G_{3,i}(t).$$

### 3. Error Estimation

The error estimation implemented in this research adopts similar strategies as suggested in [14]. Estimation of the local error for each step size begins with the predictor. The predictor can be written in the form of

$$pr y_{n+3}^{(d-j)} = \sum_{i=0}^{j-1} \frac{(3h)^i}{i!} y_n^{(d-j+i)} + h^j \sum_{i=0}^{k-1} \sigma_{3,j,i} \nabla^i f_n, \quad j = 0, 1, \dots, d.$$

Using a Predict - Evaluate - Correct - Evaluate ( $P_k EC_{k+1} E$ ) algorithm, the corrector can be formulated as follows

$$cr y_{n+3}^{(d-j)} = \sum_{i=0}^{j-1} \frac{(3h)^i}{i!} y_n^{(d-j+i)} + h^j \sum_{i=0}^{k-1} \sigma_{3,j,i}^* \nabla^i f_{n+3}.$$

For computational purpose, the corrector is then rewritten in terms of the predictor using the relationship between explicit and implicit integration coefficients, thus

$$cr y_{n+3}^{(d-j)} = pr y_{n+3}^{(d-j)} + h^j \sigma_{3,j,i}^* \nabla^k pr f_{n+3}.$$

By means of Milne error estimate, the local truncation error (LTE) is simply derived by obtaining the difference between order  $k$  and  $k - 1$ , providing the following estimate

$$Err_k^j = h^j \sigma_{3,j,i}^* \nabla_{pr}^k f_{n+3} \quad j = 0, 1, \dots, d.$$

For a suitable selection of the LTE to control the order and step size, [3] discusses the selection strategies and criteria in detail. To summarize his suggestions, a suitable LTE is determined by the appropriate selection of  $p$ , for

$$Err_k^{(d-p)} = h^{d-p} \sigma_{3,d-p,i}^* \nabla_{pr}^k f_{n+3}, \quad p = 0, 1, \dots, d.$$

#### 4. Order and Step Size

When implementing a variable order step size algorithm, the acceptance of an integration step is crucial. The efficiency and reliability of the algorithm is taken in account. The efficiency of a variable order step size algorithm is influenced by the order selection strategy whereas the reliability reflects the acceptance criteria used.

The order and step size selection is based on the local accuracy requirement in which we take

$$\theta_{n+1}^{pr} \left| Err_k^{(d-p)} \right| < TOL,$$

where  $TOL$  represents the selected tolerance and  $\theta_{n+1}^{pr}$  can be denoted as

$$\theta_{n+1}^{pr} = \frac{1}{(A + B + P_n)}.$$

The absolute error test selected when  $A = 1$ ,  $B = 0$  whereas the relative error test is selected by setting  $A = 0$ ,  $B = 1$  and when  $A = B = 1$  the mixed error test is chosen. Variable order strategies for a multistep method simply depends on the back values stored. The order can be increased if the previous back values remains and can be decreased simply by discarding the appropriate amount of back values. Literature shows that there many strategies for implementing variable order algorithm in an Adam based code. In this research, similar strategies suggested by [15] was adopted. The variable step size strategy chosen for this research adopted the step size changing technique from [15] coupled with the doubling and halving step size algorithm derived in [1].

**Table 1.** Test problems.

No.	Problem	Initial conditions	Exact solution
1	$y''(t) + 2y'(t) + y(t) + 8y^3(t) = e^{-3t}$ $0 \leq t \leq 100$	$y(0) = \frac{1}{2}$ $y'(0) = -\frac{1}{2}$	$y(t) = \frac{1}{2}e^{-t}$ Source: [16]
2	$y''(t) + y'(t) + y(t) + y(t)y^2(t) = 2 \cos t - \cos^3 t$ $0 \leq t \leq 100$	$y(0) = 0$ $y'(0) = 1$	$y(t) = \sin t$ Source: [17]
3	$y''(t) + y(t) + y^3(t) = 0$ $0 \leq t \leq 5$	$y(0) = 1$ $y'(0) = 0$	none Source: [18]
4	$y''''(t) + 5y''(t) + 4y(t) - \frac{1}{6}y^3(t) = 0$ $0 \leq t \leq 5$	$y(0) = 0$ $y'(0) = 1.91103$ $y''(0) = 0$ $y'''(0) = -1.15874$	$y(t) = 2.1906 \sin 0.9t$ $-0.02247 \sin 2.7t$ $-0.000045 \sin 4.5t$ Source: [19]

**5. Numerical Simulations and Discussions**

Numerical method for solving Duffing oscillator can found in research by authors such as [16–20] and many others. Numerical results of the proposed Three-Point Block Variable Order Step Size (3PBVOS) algorithm was compared against other multistep methods and established conventional methods. Test Problem 1 and 2 consists of non-homogeneous second order ODEs whereas test Problem 3 and 4 are of homogeneous ODEs. Problem 3 is a well-known Duffing Oscillator without any known exact solution. Finally, Problem 4 is a higher order (fourth order) nonlinear ODE which was added to include a test problem with a certain level of difficulty. For the purpose of simplicity, the following are abbreviation will be used in the current section

ERR: the overall maximum error,  
 AVE: the average error,  
 MTHD: the method used,  
 STEPS: total steps,  
 TOL: tolerance,  
 SN: standard numerical,

VOSBD: VOS backward difference,  
 2PBVOS: 2-Point Block Variable Order Stepsize,  
 3PBVOS: 3-Point Block Variable Order Stepsize,  
 SHP: standard homotopy perturbation,  
 DI: Direct Integration.

Tables 2-3 and Figures 1-2 shows the competitive nature of the 3PBVOS against VOSBD, 2PBVOS and DI methods. Numerical results also show the consistency of the proposed methods in contrast to the other methods, where there are sudden increases or decreases in terms of step size and accuracy at certain tolerances.

**Table 2.** Numerical Results for Problem 1.

TOL	MTHD	STEP	ERR	AVE
10 <sup>-2</sup>	DI	156	4.5415(-2)	1.1292(-2)
	VOSBD	154	5.5442(-2)	9.0456(-3)
	2PBVOS	219	9.9705(-1)	1.1201(-1)
	3PBVOS	324	1.8129(-1)	7.2336(-2)
10 <sup>-4</sup>	DI	169	3.2896(-4)	8.4885(-5)
	VOSBD	215	4.9022(-4)	6.4746(-5)
	2PBVOS	150	2.7777(-3)	1.6877(-4)
	3PBVOS	321	1.3584(-3)	4.3135(-4)
10 <sup>-6</sup>	DI	173	2.1953(-5)	2.8045(-6)
	VOSBD	236	1.3169(-5)	1.9170(-6)
	2PBVOS	160	1.1849(-5)	1.3204(-6)
	3PBVOS	333	1.9814(-5)	2.2591(-6)
10 <sup>-8</sup>	DI	204	1.4889(-7)	3.1000(-8)
	VOSBD	224	1.8241(-7)	2.2270(-8)
	2PBVOS	192	1.9583(-7)	2.7345(-8)
	3PBVOS	347	1.5230(-7)	6.7281(-8)
10 <sup>-10</sup>	DI	317	1.1856(-9)	3.9460(-10)
	VOSBD	224	1.1245(-9)	2.0484(-10)
	2PBVOS	217	2.8051(-9)	1.1772(-9)
	3PBVOS	376	4.9294(-9)	8.2086(-10)

**Table 3.** Numerical Results for Problem 2.

TOL	MTHD	STEP	ERR	AVE
10 <sup>-2</sup>	DI	254	8.4907(-2)	2.0479(-2)
	VOSBD	217	1.0760(-1)	3.0389(-2)
	2PBVOS	163	5.9043(-1)	6.8756(-2)
	3PBVOS	218	7.9247(-1)	2.9491(-2)
10 <sup>-4</sup>	DI	332	1.5470(-3)	4.7263(-4)
	VOSBD	284	1.2464(-3)	1.9289(-4)
	2PBVOS	186	3.1849(-3)	7.3645(-4)
	3PBVOS	243	3.7669(-3)	6.3356(-4)
10 <sup>-6</sup>	DI	382	4.2408(-5)	1.5684(-6)
	VOSBD	330	1.2803(-5)	3.2664(-6)
	2PBVOS	279	1.2298(-5)	3.8555(-6)
	3PBVOS	373	1.3830(-4)	9.2963(-6)
10 <sup>-8</sup>	DI	651	7.9360(-7)	2.0372(-7)
	VOSBD	499	7.2732(-7)	1.0138(-7)
	2PBVOS	609	3.1981(-7)	1.9188(-8)
	3PBVOS	560	2.1072(-6)	1.7452(-7)
10 <sup>-10</sup>	DI	772	1.1856(-9)	2.0372(-9)
	VOSBD	702	1.1245(-9)	1.0138(-9)
	2PBVOS	1376	2.8051(-9)	1.2811(-10)
	3PBVOS	943	1.0193(-8)	8.6959(-10)

As previously mentioned, Test Problem 3 is a Duffing oscillator without any known exact solution. Comparison obtained from Table 4 confirms the accuracy of 3PBVOS method. Results show that 3PBVOS provides the same approximated solution as other established methods. Finally, Table 5 and Figure 4 entails numerical simulation of four variable order step size algorithm (DI, VOSBD, 2PBVOS and 3PBVOS) for solving a fourth order Duffing type oscillator. In this table, numerical results clearly show the advantages of the 3PBVOS algorithm. The 3PBVOS method is able to maintain the same accuracy at the same set of tolerance but with the least amount of time steps required. The efficiency of a

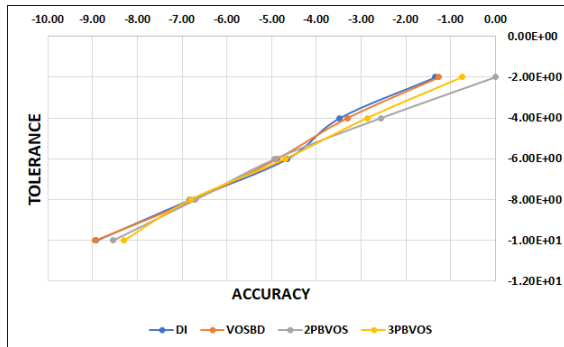


Figure 1. Accuracy of DI, VOSBD and 2PBVOS method for Problem 1.

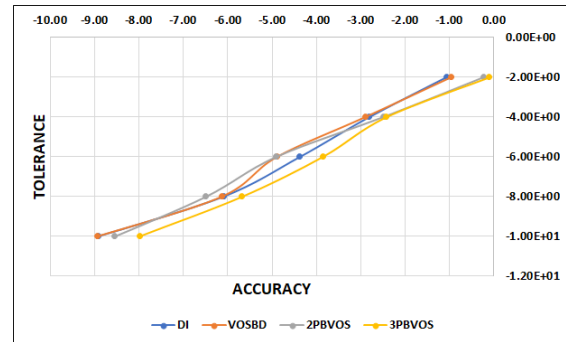


Figure 2. Accuracy of DI, VOSBD and 2PBVOS method for Problem 2.

variable order step size algorithm is commonly defined by total steps required per accuracy or graphical terms, the undermost curve. As illustrated in Figure 4, the 3PBVOS presents the undermost curve hence, by definition is proven to be the most efficient out of all four algorithms.

Table 4. Numerical Results for Problem 3.

$x$	VOSBD	2PBVOS	3PBVOS	SHPM	SNM
	TOL = $1 \times 10^{-10}$				
0.5	7.68802(-1)	7.68802(-1)	7.68802(-1)	7.68766(-1)	7.68802(-1)
1.0	2.33692(-1)	2.33692(-1)	2.33692(-1)	2.33680(-1)	2.33692(-1)
2.0	-8.59349(-1)	-8.59349(-1)	-8.59349(-1)	-8.9323(-1)	-8.59349(-1)
3.5	-9.30130(-1)	-9.30130(-2)	-9.30130(-2)	-9.30340(-2)	-9.30130(-2)
5.0	9.47130(-1)	9.47130(-1)	9.47130(-1)	9.47107(-1)	9.47130(-1)

Table 5. Numerical Results for Problem 4.

TOL	MTHD	STEP	ERR	AVE
$10^{-2}$	DI	48	2.0630(-1)	3.7520(-2)
	VOSBD	46	2.1990(-2)	4.5935(-3)
	2PBVOS	44	6.6842(-2)	1.5850(-2)
	3PBVOS	31	4.6466(-1)	6.3490(-2)
$10^{-4}$	DI	80	6.7457(-4)	2.4460(-4)
	VOSBD	86	1.5828(-3)	3.9442(-4)
	2PBVOS	49	3.2871(-4)	6.4011(-5)
	3PBVOS	47	2.2736(-3)	5.0798(-4)
$10^{-6}$	DI	139	1.3772(-4)	1.5684(-5)
	VOSBD	102	1.2637(-4)	3.2664(-5)
	2PBVOS	87	2.4962(-4)	3.8555(-5)
	3PBVOS	62	1.2358(-4)	2.9781(-5)
$10^{-8}$	DI	388	1.2815(-4)	1.5684(-5)
	VOSBD	126	1.2309(-4)	3.2664(-5)
	2PBVOS	114	1.2905(-4)	3.8555(-5)
	3PBVOS	126	1.2949(-4)	3.9812(-5)
$10^{-10}$	DI	308	1.2701(-4)	1.5684(-5)
	VOSBD	225	1.1954(-4)	3.2664(-5)
	2PBVOS	236	1.2889(-4)	3.8555(-5)
	3PBVOS	211	1.2880(-4)	4.2460(-5)

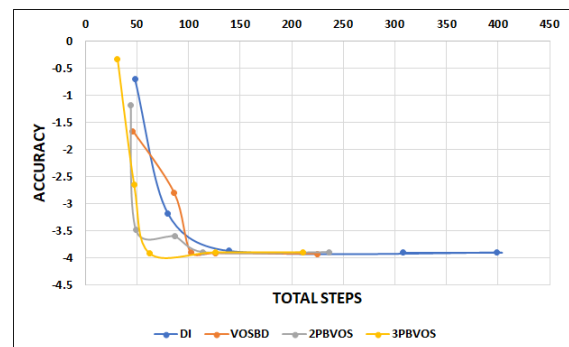


Table 6. Accuracy of DI, VOSBD and 2PBVOS method for Problem 4.

Numerical results as illustrated in tables and figures above clearly supports the proposed 3PBVOS method as a viable alternative for solving Duffing type ODEs.

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