CONFERENCE PROCEEDING

Numerical Solution of Homogeneous One-Dimensional Heat Equation Using Crank-Nicolson Method

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Abstract

A partial differential equation is an equation which includes derivatives of an unknown function with respect to two or more independent variables. The numerical solution is needed to obtain the solution of partial differential equation. To solve these partial differential equations, the appropriate boundary and initial conditions are needed. The general solution is dependent not only on the equation, but also on the boundary conditions. In other words, these partial differential equations will have different general solution when paired with different sets of boundary conditions. In the present study, the homogeneous one-dimensional heat equation will be solved numerically by using Implicit Crank Nicolson method. Our main objective is to determine the flow characteristics of heat equation with Dirichlet boundary condition on homogeneous heat equation. The method of Implicit Crank Nicolson has been chosen because of the stability of the method. The results have been compared with the exact analytical solution. The validated results show that the numerical results remains same as the exact analytical solutions. The results show that the changes of the temperature profile depends on the types of boundary conditions. The boundary conditions will be affected the flow characteristics of the heat equation.

Keywords: *GENIUS INAQ, heat equation, homogeneous one-dimensional, Dirichlet boundary conditions, Implicit Crank-Nicolson method*

INTRODUCTION

A partial differential equation is an equation that involves derivatives of an unknown function with respect to two or more independent variables. It is possible to classify the partial differential equation into three form which are parabolic (Agyeman & Folson, 2003; Mamun *et al.,* 2018), hyperbolic and elliptic (Biala & Jator, 2015; Papanikos & Gousidou-Koutita, 2015). The heat equation propagates energy, which is highly non-physical, at an infinite speed and it defined as a process of energy transfer as a result of temperature difference between the two points (Abdulla *et al*., 2018). However, for all classical physics and engineering applications, the validity of the heat equation as a model of temperature evolution is still extremely strong. Temperature fluctuations are one of the main consequences of heat transfer, where the heating process raises the temperature while the cooling process reduces the temperature (Tveito & Winther, 1998).

Dirichlet, Neumann and Robins are three types of boundary conditions of second order linear partial differential equations (Zill & Cullen, 2009), where Dirichlet, defined with a temperature function, *u*, Neumann defined as the (spatial) derivative of the temperature function, *u*, while Robins, defined as a linear combinations of velocity and its derivatives. According to Subani *et al*. (2020), a given partial differential equation can be solved by using numerical solution and analytical solution. However, to ensure that the numerical solution is valid, it is important to understand the general theory of partial differential equations. (Abarbanel *et al*., 2000; Islam *et al*., 2018; Mebrate, 2015; Roknujjaman & Asaduzzaman, 2018). According to Islam *et al*. (2018), Crank Nicolson method is one of the numerical method that is used to find the solution of one-dimensional heat equation. This method always dealing with complex problems of science and technology.

The suitable boundary and initial conditions are required to solve these partial differential equations. Not only the equation, but also the boundary conditions are depending on the general solution. In other words, when combined with various sets of boundary conditions, these partial differential equations would have distinct general solutions. However, Subani *et al*. (2020) only consider the heat equation with Neumann boundary condition, and they found that the heat temperature quickly converges to zero compared to the long rod.

Sabaeian *et al.* (2008) claimed that the in Islamic perspective, that the role of temperature distribution is important in the measurement, simulation, and prediction of thermal effects. Temperature is specific to heat capacity, or the quantity of energy needed to adjust the substance's temperature. The measuring heat changes due to physical or chemical changes (Aidoo & Wilson, 2015).

According to Al-Mahalli (2003) and As-Suyuti (2003), the verse in *Quran* (*Surah Yassin*: 80) tells us about the development of the fire from green trees. In other words, by using green plants, fire can be made. In that life, the friction of two surface objects generates fire (Al-Mahalli & As-Suyuti, 2007). In this research, the heat velocity would be calculated from one zone of high to low heat. The rate of heat velocity is dependent upon the degree of friction velocity between the two objects.

In the present research, the homogeneous one-dimensional heat equation will be solved numerically by using Implicit Crank Nicolson method. Our main objective is to determine the flow characteristics of heat equation with difference types of boundary conditions on homogeneous heat equation. To verify our objective, the heat equation will be solved based on Dirichlet boundary conditions. The method of Implicit Crank Nicolson has been chosen because of the stability of the method. The validated results show that the numerical results remains same as the exact analytical solutions. The results have been compared with different values of boundary conditions but the initial condition remain the same.

MATHEMATICAL FORMULATION

The mathematical models are used to describe the one-dimensional homogeneous heat boundary value problems with Dirichlet boundary conditions are presented below. The heat equation is used to determine the change in the function of temperature, *u* over time, *t*. The physical model of the heat equation problem is shown in Figure 1.

$u(x,0) = f(x)$

(Initial temperature

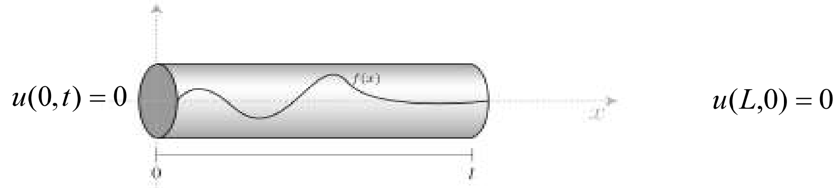

Figure 1. Physical model of heat equation problem with Dirichlet boundary condition

Boundary Value Problem

The partial differential equation of one-dimensional homogeneous heat conduction equation is given by:

$$u_t(x,t)=cu_{xx}(x,t),\quad 0<x<L,\quad t>0 \qquad (1)$$

where u is defined as heat temperature, x is space, t is time, c is thermal diffusivity and L is the end length of the rod.

Boundary and Initial Conditions

The Dirichlet boundary conditions at the initial point, $x=0$ and at the end point $x=L$ are given by:

$$u(0,t)=T_o,\quad t>0 \text{ and } u(L,t)=T_i,\quad t>0 \qquad (2a)$$

where T_o is a constant temperature along time axis, T_i is a constant temperature along $x=L$.

The initial conditions at $t=0$ is:

$$u(x,0)=f(x),\quad 0<x<L \qquad (3)$$

The heat conduction in a single-dimensional uniform rod of length one unit without internal heat sources, thermal diffusivity *c*, perfect lateral insulation, the temperature is kept at $0^{\circ}C$ and initial conditions *f*(*x*) when $0<x<L$ were defined by these mathematical models of equations (1)-(3). To solve the equation (1), the heat equation should be written in the form of u_j^i. At the point u_j^i the finite difference approximation for $u_{xx}(x.t)$ is:

$$u_{xx}(x,t)=\frac{u_{j+1}^{i+1}-2u_{j+1}^{i}+u_{j+1}^{i-1}}{h^2} \qquad (4)$$

NUMERICAL SOLUTION

Using Crank-Nicolson (Emenogu & Oko, 2015), finite different method (Gerald, 2011; Jalil, 2011; Zana, 2014) or finite element method (Dabral *et al*., 2011; Susan *et al*., 2008), most of the heat equation will be solved numerically. In this research, the Crank-Nicolson method will be used to solve the homogeneous heat equation. Crank-Nicolson method is an interesting second order implicit scheme can be obtained by averaging the forward time central-space and the backward time central-space scheme. This method was inverted by John Crank and Phyllis Nicolson. The originally applied it to the heat equation and they approximated the solution of the heat equation on same words, the Crank-Nicolson method is numerically stable and it only requires the solution of a very simple system of linear equations at every time level.

However, this Crank-Nicolson method had some disadvantages. It requires the inverse of a matrix to be calculated and expense operation to perform. Another issue with implicit Crank-Nicolson method is that his method is very sensitive to non-smooth boundary conditions. A modified approach using the explicit method for the first few time steps then reverting to the implicit Crank-Nicolson method for the remaining steps often overcomes this problem. Therefore, in this research, the implicit Crank-Nicolson is used to solve the heat problem. This is because, the Implicit Crank-Nicolson scheme is a stable method. It is of higher order accuracy $O(\Delta t^2, \Delta x^2)$ and involves a little more work in solving each update but is more accurate than the implicit scheme can be pretty complicated to implement but it is one of the preferred techniques problems like the one dimensional parabolic equation.

Implicit Crank-Nicolson Method

By using the Implicit Crank-Nicolson method, the heat equation (1) can be written as:

$$\frac{u_{j+1}^{i} - u_{j}^{i}}{2\left(\frac{k}{2}\right)} = \frac{v}{2}\left(\frac{u_{j+1}^{i+1} - 2u_{j+1}^{i} + u_{j+1}^{i-1}}{h^2} + \frac{u_{j}^{i+1} - 2u_{j}^{i} + u_{j}^{i-1}}{h^2}\right) \quad (5)$$

$$2\left(u_{j+1}^{i} - u_{j}^{i}\right) = \frac{vk}{h^2}\left(u_{j+1}^{i+1} - 2u_{j+1}^{i} + u_{j+1}^{i-1} + u_{j}^{i+1} - 2u_{j}^{i} + u_{j}^{i-1}\right) \quad (6)$$

By assuming $r = \frac{vk}{h^2}$, the equation (6) can be rewritten as:

$$-ru_{j+1}^{i-1} + (2+2r)u_{j+1}^{i} - ru_{j+1}^{i+1} = ru_{j}^{i-1} + (2-2r)u_{j}^{i} + ru_{j}^{i+1} \quad (7)$$

Now, setting $r = \frac{vk}{h^2} = \frac{2(0.02)}{0.4^2} = 1$, then the equation (7) yields:

$$-u_{j+1}^{i-1}+4u_{j+1}^{i}-u_{j+1}^{i+1}=u_{j}^{i-1}+u_{j}^{i+1} \tag{8}$$

The computational molecule of Crank-Nicolson method and the calculation will be done by using Matlab software have been illustrated in Figure 2.

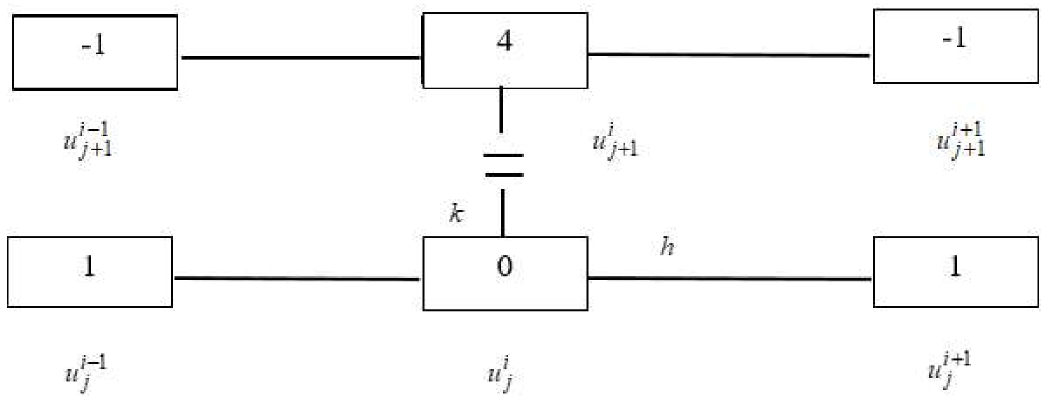

Figure 2. Step calculation of Implicit Crank-Nicolson method

Analytical Method

In this research, the analytical solution of heat equation (1) have been solved analytically. Equation (9) are used to compared the exact solution with the numerical method which is Implicit Crank-Nicolson method.

$$u(x,t)=\sum_{n=1}^{\infty}\frac{2}{n\pi}\left[1-(-1)^{n}\right]\sin(n\pi x)\,e^{-(n\pi)^{2}t} \tag{9}$$

RESULTS AND DISCUSSION

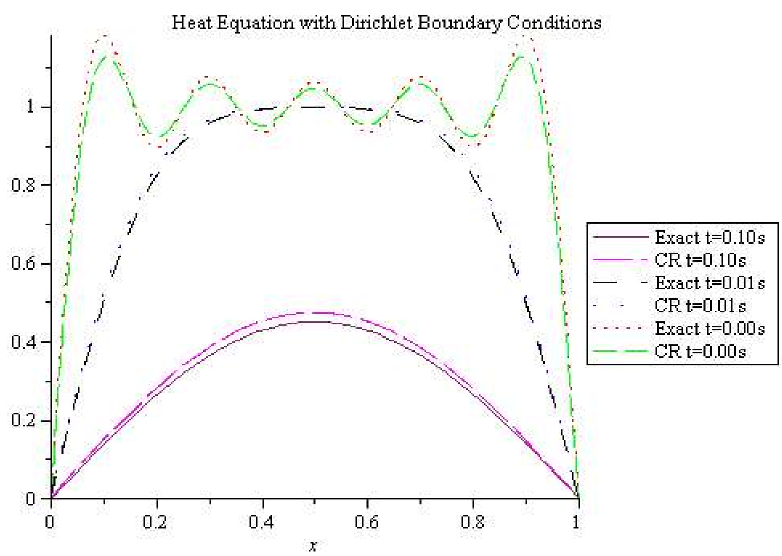

Figure 3. Validated results on Dirichlet boundary condition using Implicit Crank-Nicolson method

Figure 3 show the heat profile along the rod at $t > 0$ for Dirichlet boundary conditions of homogeneous heat equation. The times are varying at 0.00s, 0.01s and 0.10s, at number of iteration, ∞ and the length of the rod, *L*=1m. The results in Figure 3 are validated by using the method of Crank-Nicolson. The validated results are compared with analytical solution where the exact solution are calculated in equation (9). Based on the Figure 3, the validated results are very precise or remain the same as the exact solutions. Thus, this Implicit Crank-Nicolson method is the best method to solve the heat equation.

CONCLUSION

The numerical solution is used to obtain the solution of partial differential equation. To solve these partial differential equations, the appropriate boundaries and initial conditions are required. The general solution depends on not only the equation, but also the boundary conditions. In particular, these partial differential equations will have different general solutions if combined with differing sets of boundary conditions. In the current analysis, the homogeneous one-dimensional heat equation will be solved numerically by using Implicit Crank Nicolson method. Our primary aim is to determine the flow characteristics of heat equation with Dirichlet boundary condition on homogeneous heat equation. The times are varying at 0.00s, 0.01s and 0.10s, at number of iteration, $1 \leq n \leq \infty$ and the length of the rod, *L*=1m. The validated results of the Implicit Crank-Nicolson method are compared with analytical solution where the exact solution. The finding results are very precise or remain the same as the exact solutions. The method of Implicit Crank Nicolson has been chosen because of the stability of the method. Thus, this Implicit Crank-Nicolson method is the best method to solve the heat equation.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests regarding the publication of this paper.

ACKNOWLEDGEMENT

Kolej GENIUS Insan and Universiti Sains Islam Malaysia, are gratefully acknowledged.

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