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Extended Chebyshev Polynomials for Solving Bounded and Unbounded Singular Integral Equations

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Abstract. In this note, we have developed new classes of one dimensional orthogonal polynomials $Z_{(i,n)}^k(x), i = \{1, 2\}, n = 0, 1, 2, \dots$, namely extended Chebyshev polynomials (ECPs) of the first and second kinds, which are an extension of the Chebyshev polynomials of the first and second kinds respectively. For non-homogeneous SIEs (bounded and unbounded case) truncated series of the first and second kind of ECPs are used to find approximate solution. It is found that first and second kinds of ECPs $Z_{(i,n)}^k(x), i = \{1, 2\}$ are orthogonal with weights $w_{(1,k)}(x) = \frac{x^{k-1}}{\sqrt{1-x^{2k}}}$ and $w_{(2,k)}(x) = x^{k-1}\sqrt{1-x^{2k}}$, where k is positive odd integer. Spectral properties of first and second kind of ECPs are also proved. Finally, two examples are presented to show the validity and accuracy of the proposed method.

1. Introduction

Many phenomena of almost all practical applications, such as physical systems, economics, engineering, electrical network analysis and medicine are represented to the mathematical models, some of these models can be formulated as integral equations. Since, in many cases, integral equations can not be solved analytically, it is required to obtain approximate solutions. There are many different methods to find approximate solution of integral equations [1]-[3]. One of the effective approximate method is to use orthogonal function to represent the unknown time functions. The main characteristic of this technique is to reduce integral equations into the systems of algebraic equations, which is easy to solve. Orthogonal functions method have been proposed to solve linear integral equations of the first and second kind, particularly, applications of Walsh functions [4], block-pulse functions [5], Legendre polynomials [6], Laguerre-Gaussians quadrature formula [7], Chebyshev polynomials [8]. An important type of integral equation that contains a singular kernel, many researchers have proposed different methods to solve singular integral equations (SIEs) and hyper-singular integral equations (HSIEs), approximately [9]-[12].

In the present paper, we have defined a new class of orthogonal polynomials of the first and second kinds named ECPs $Z_{(i,n)}^k(x), i = \{1, 2\}$, where k is odd positive integer with $x \in [-1, 1]$ and used it to solve:

$$\int_{-1}^1 \frac{f(x)}{t^k - x^k} dx = g(t^k), \quad t \in (-1, 1) \quad (1)$$

where k is a positive odd integer. The method presented here consists of expanding the unknown



solution $f(x)$ by ECPs $Z_{(i,n)}^k(x)$ with unknown coefficients and main problem is to find those unknown coefficients.

The structure of the note is arranged as follows: in Section 2, we provide some theoretical aspects of inner product (Hilbert) space and prove Lemmas. In section 3, derivation of the proposed method for solving (1) for bounded and unbounded cases are presented. Numerical examples are provided in Section 4. Finally, conclusion and acknowledgment are given in Section 5.

2. Preliminaries

2.1. Orthogonal polynomials and weight functions

Definition 1 Two functions $f(x)$ and $g(x)$ in $L_2[a, b]$ are said to be orthogonal on the interval $[a, b]$ with respect to a weight function $w(x)$ if

$$\langle f, g \rangle_w = \int_a^b w(x)f(x)g(x)dx = 0. \quad (2)$$

The norm of f in inner product space L_2 is defined as

$$\|f\| = \|f\|_2 = \sqrt{\langle f, f \rangle}. \quad (3)$$

Definition 2 (Mason [13]) Chebyshev polynomials of the first kind $T_n(x)$ and second kind $U_n(x)$ are the polynomials in x of degree n , defined by the relationship

$$T_n(x) = \cos(n\theta), \quad x = \cos(\theta), \quad n = 0, 1, 2, \dots \quad (4)$$

$$U_n(x) = \frac{\sin((n+1)\theta)}{\sin\theta}, \quad x = \cos(\theta), \quad n = 0, 1, 2, \dots \quad (5)$$

and it forms an orthogonal system on $[-1, 1]$ with respect to the weights $w_1(x) = \frac{1}{\sqrt{1-x^2}}$ and $w_2(x) = \sqrt{1-x^2}$, respectively

$$\langle T_n, T_m \rangle = \int_{-1}^1 \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}} dx = \begin{cases} 0, & m \neq n, \\ \frac{\pi}{2}, & m = n, \\ \pi, & m = n = 0. \end{cases} \quad (6)$$

$$\langle U_n, U_m \rangle = \int_{-1}^1 \sqrt{1-x^2} U_n(x)U_m(x) dx = \begin{cases} 0, & m \neq n, \\ \frac{\pi}{2}, & m = n. \end{cases} \quad (7)$$

Definition 3 The extended Chebyshev polynomials of the first kind $Z_{(1,n)}^k(x)$ and second kind $Z_{(2,n)}^k(x)$ of order k , where k is positive odd integer number, are the polynomials in x of degree (kn) , defined by the relationship (for fixed k)

$$Z_{(1,n)}^k(x) = \cos(n\theta), \quad x^k = \cos(\theta), \quad (8)$$

$$k = 1, 3, 5, \dots, 2N-1, \quad n = 0, 1, 2, \dots$$

$$Z_{(2,n)}^k(x) = \frac{\sin((n+1)\theta)}{\sin\theta}, \quad x^k = \cos(\theta), \quad (9)$$

$$k = 1, 3, 5, \dots, 2N-1, \quad n = 0, 1, 2, \dots$$

The range of the variable x is $[-1, 1]$ and the range of the corresponding variable θ is $[0, \pi]$. The above definition is well defined and for $x \in [-1, 1]$ with $|\cos(\theta)| \leq 1$, implies $|x|^k \leq 1$.

The relationship between Extended Chebyshev polynomials (ECPs) and Chebyshev polynomials are:

$$Z_{(i,n)}^k(x) = \begin{cases} T_n(x^k), & i = 1, \\ U_n(x^k), & i = 2, \end{cases} \quad (10)$$

where $k > 1$ odd positive integers and $T_n(x), U_n(x)$ are Chebyshev polynomials of the first and second kinds respectively and $n = 0, 1, 2, 3, \dots$. For a case of $k = 1$, ECPs coincides with standard Chebyshev polynomials.

In Table 1 few terms of ECPs of the first and second kinds of order k is represented.

In the next subsection we do state some relationship between Chebyshev polynomials and the ECPs of the first and second kind of order k .

2.2. Properties of Extended Chebyshev Polynomials

Let interval $[a, b] = [-1, 1]$ and weight functions be defined by

$$w_{(i,k)}(x) = \begin{cases} x^{k-1}, & i = 1, \\ \frac{x^{k-1}}{\sqrt{1-x^{2k}}}, & i = 2. \end{cases} \quad (11)$$

where k is positive odd integers, then the inner product ECPs of the first and second kinds $Z_{(1,n)}^k(x)$ is defined as

$$\langle Z_{(i,n)}^k, Z_{(i,m)}^k \rangle_{w_{(i,k)}} = \int_{-1}^1 w_{(i,k)} Z_{(i,n)}^k(x) Z_{(i,m)}^k(x) dx, \quad i = \{1, 2\}. \quad (12)$$

$n, m = 0, 1, 2, 3, \dots$

Lemma 2.1 (Berthold et al. [14])

$$\int_0^\pi \frac{\cos(n\theta)}{\cos(\theta) - \cos(\phi)} d\theta = \pi \frac{\sin(n\phi)}{\sin(\phi)}, \quad (13)$$

for any $\phi \in [0, \pi]$, $n = 0, 1, 2, 3, \dots$

Lemma 2.2 (Berthold et al. [14])

$$\int_0^\pi \frac{\sin(n\theta) \sin(\theta)}{\cos(\theta) - \cos(\phi)} d\theta = -\pi \cos(n\phi), \quad (14)$$

for any ϕ in $[0, \pi]$, $n=1, 2, 3, \dots$

Lemma 2.3 Let $Z_{(1,n)}^k(x)$ be an ECPs of the first kind then

$$(i) Z_{(1,n)}^k(x) = T_n(x^k) = \cos(n \cos^{-1}(x^k)) = \cos(n\theta), \quad k = 1, 3, 5, \dots, \quad n = 0, 1, 2, \dots$$

(ii) The three terms recurrence relations of the first kind is

$$Z_{(1,n)}^k(x) = 2x^k Z_{(1,n-1)}^k(x) - Z_{(1,n-2)}^k(x), \quad Z_{(1,0)}^k(x) = 1, \quad Z_{(1,1)}^k(x) = x^k. \quad (15)$$

(iii) For each $t \in (-1, 1)$

$$\int_{-1}^1 w_{(1,k)}(x) \frac{Z_{(1,n)}^k(x)}{x^k - t^k} dx = \frac{\pi}{k} Z_{(2,n-1)}^k(t), \quad n = 1, 2, \dots; \tag{16}$$

Particularly,

$$\int_{-1}^1 w_{(1,k)}(x) \frac{Z_{(1,0)}^k(x)}{x^k - t^k} dx = 0. \tag{17}$$

(iv) The set of ECPs of the first kind of order k , i.e. $Z_{(1,n)}^k(x)$, $k = 1, 3, \dots$, $n = 0, 1, 2, \dots$ is a set of orthogonal function over $[-1, 1]$ with respect to the weight function $w_{(1,k)}(x)$ defined by (11) i.e.

$$\begin{aligned} &\langle Z_{(1,n)}^k, Z_{(1,m)}^k \rangle_{w_{(1,k)}} \\ &= \int_{-1}^1 \frac{x^{k-1} Z_{(1,n)}^k(x) Z_{(1,m)}^k(x)}{\sqrt{1-x^{2k}}} dx = \begin{cases} 0, & n \neq m, \\ \frac{\pi}{2k}, & n = m \neq 0, \\ \frac{\pi}{k}, & n = m = 0. \end{cases} \end{aligned}$$

Proof (i) The proof directly comes by letting $u = x^k$.

(ii) Using the trigonometric identity $\cos(\theta) \cos(n\theta) = \cos(n+1)\theta - \cos(n-1)\theta$, and using Definition 3 with $x^k = \cos\theta$ we arrive the results.

(iii) Using the Definition 3 for $Z_{(1,n)}^k(x)$ with corresponding weight function $w_{(1,k)}(x)$ and letting $x^k = \cos\theta \Rightarrow kx^{k-1}dx = -\sin\theta d\theta$ with $x = -1 \Rightarrow \theta = \pi$, and $x = 1 \Rightarrow \theta = 0$, and implementing results of Lemma 2.1 we arrive at

$$\begin{aligned} I(Z_{1,n}^k) &= \int_{-1}^1 w_{(1,n)}(x) \frac{Z_{(1,n)}^k(x)}{x^k - t^k} dx \\ &= \frac{1}{k} \int_0^\pi \frac{\cos n\theta}{\cos\theta - \cos\phi} d\theta = \frac{\pi \sin(n\phi)}{k \sin\phi} = \frac{\pi}{k} Z_{2,n-1}^k(t), \end{aligned}$$

where $n = 1, 2, 3, \dots$, $k = 1, 3, \dots$ and $t^k = \cos\phi \in (-1, 1)$.

Particularly, for $n = 0$ using the results of Israilov [?] we have

$$\begin{aligned} I(Z_{1,0}^k) &= \int_{-1}^1 w_{(1,k)}(x) \frac{Z_{(1,0)}^k(x)}{x^k - t^k} dx = \frac{1}{k} \int_{-1}^1 \frac{du}{\sqrt{1-u^2}(u-\eta)} \\ &= \frac{1}{k\sqrt{1-\eta^2}} \ln \left| \frac{u\sqrt{1-\eta^2} - \eta\sqrt{1-u^2}}{\sqrt{1-\eta^2} + \sqrt{1-u^2}} \right|_{-1}^1 = 0, \end{aligned}$$

where $u = x^k$, $\eta = t^k$ and $Z_{(1,0)}^k(x) = 1$.

(iv) Using the Definition 3 and letting $x^k = \cos\theta \Rightarrow kx^{k-1}dx = -\sin\theta d\theta$, we obtain

$$\begin{aligned} \langle Z_{(1,n)}^k, Z_{(1,m)}^k \rangle_{w_{(1,k)}} &= \int_{-1}^1 \frac{x^{k-1} Z_{(1,n)}^k(x) Z_{(1,m)}^k(x)}{\sqrt{1-x^{2k}}} dx = \frac{1}{k} \int_0^\pi \cos(n\theta) \cos(m\theta) d\theta \\ &= \frac{1}{2} \left(\frac{\sin(n+m)\theta}{n+m} - \frac{\sin(n-m)\theta}{n-m} \right)_0^\pi = \begin{cases} 0, & n \neq m, \\ \frac{\pi}{2k}, & n = m \neq 0, \\ \frac{\pi}{k}, & n = m = 0. \end{cases} \end{aligned}$$

Lemma 2.4 Let $Z_{(2,n)}^k(x)$ be an ECPs of the second kind then

$$(i) \quad Z_{(2,n)}^k(x) = U_n(x^k) = \frac{\sin((n+1)\cos^{-1}(x^k))}{\sin(\cos^{-1}(x^k))} = \frac{\sin(n+1)\theta}{\sin\theta},$$

$$k = 1, 3, 5, \dots, \quad n = 0, 1, 2, \dots$$

(ii) The three terms recurrence relations of the second kind is

$$Z_{(2,n)}^k(x) = 2x^k Z_{(1,n-1)}^k(x) - Z_{(1,n-2)}^k(x), \quad Z_{(2,0)}^k(x) = 1, \quad Z_{(2,1)}^k(x) = 2x^k. \quad (18)$$

(iii) For each $t \in (-1, 1)$

$$\int_{-1}^1 w_{(2,k)}(x) \frac{Z_{(2,n-1)}^k(x)}{x^k - t^k} dx = -\frac{\pi}{k} Z_{(1,n)}^k(t), \quad n = 1, 2, \dots \quad (19)$$

(iv) The set of ECPs of the second kind of order k , i.e. $Z_{(2,n)}^k(x)$, $k = 1, 3, 5, \dots$, $n = 0, 1, 2, \dots$ is a set of orthogonal function over $[-1, 1]$, with respect to the weight function $w_{(2,k)}(x)$ defined by (11) i.e.

$$\langle Z_{(2,n)}^k, Z_{(2,m)}^k \rangle_{w_{(1,k)}} = \int_{-1}^1 x^{k-1} \sqrt{1-x^{2k}} Z_{(2,n)}^k(x) Z_{(2,m)}^k(x) dx = \begin{cases} 0, & m \neq n, \\ \frac{\pi}{2k}, & m = n. \end{cases} \quad (20)$$

Proof of the Lemma 2.4 can be obtained in a similar way as the proof of Lemma 2.3 and using the Lemma 2.2 and following identity

$$2 \cos(\theta) \sin(n\theta) = \sin(n+1)\theta + \sin(n-1)\theta.$$

According to the Definition 3, ECPs of the first kind and second kind have the zeros of the form

$$x = x_{(1,m)} = \sqrt[k]{\cos \frac{(m - \frac{1}{2})\pi}{n}}, \quad m = \{1, 2, 3, \dots, n\},$$

$$x = x_{(2,m)} = \sqrt[k]{\cos \frac{m\pi}{n+1}}, \quad m = \{1, 2, 3, \dots, n+1\}.$$

3. Function Approximation

In the present section, we link with the spectral properties of the ECPs first and second kinds.

3.1. Solution of special singular integral equation of order k .

In this section, we will begin our study of special singular integral equation of order k (SSIEk), which has in form

$$\int_{-1}^1 \frac{f(x)}{t^k - x^k} dx = g(t^k), \quad t \in (-1, 1) \quad (21)$$

where k is a positive odd integer.

In particular, when $k = 1$, Eq.(21) reduces to the standard airfoil equation and it has four types of analytical solutions (for more details see Lifanov [15]). For solving Eq.(21) is based on expansions of the approximate solution in certain ECPs of the first and second kinds $Z_{(i,n)}^k(x)$, $i = \{1, 2\}$ with two cases:

Case (I). The solution is unbounded at both end-points $x = \pm 1$. For the unbounded solution of Eq.(21), we impose the following condition

$$\int_{-1}^1 f(x) dx = c, \quad c \text{ is constant.} \quad (22)$$

and $f(t)$ is approximated by

$$f_n(t) = a_0 w_{(1,k)}(t) Z_{(1,0)}^k(t) + w_{(1,k)}(t) \sum_{j=1}^n a_j Z_{(1,j)}^k(t), \quad k = 1, 3, 5, \dots \quad (23)$$

Substituting Eq. (23) into Eq. (21) yields

$$\sum_{j=0}^n \int_{-1}^1 \frac{w_{(1,k)}(x)}{x^k - t^k} a_j Z_{(1,j)}^k(x) dx = g(t^k). \quad (24)$$

Since $\int_{-1}^1 \frac{w_{(1,k)}(x)}{x^k - t^k} Z_{(1,0)}^k(x) dx = 0$, we have

$$\begin{aligned} a_1 \int_{-1}^1 \frac{w_{(1,k)}(x)}{x^k - t^k} Z_{(1,1)}^k(x) dx + a_2 \int_{-1}^1 \frac{w_{(1,k)}(x)}{x^k - t^k} Z_{(1,2)}^k(x) dx + \dots \\ + a_n \int_{-1}^1 \frac{w_{(1,k)}(x)}{x^k - t^k} Z_{(1,n)}^k(x) dx = g(t^k). \end{aligned} \quad (25)$$

Here the first coefficient a_0 is equal to zero because of condition (22) i.e $a_0 = 0$. Due to Lemma 2.3 (16), Eq. (25) can be reduced into the form

$$\frac{\pi}{k} a_1 Z_{(2,0)}^k(t) + \frac{\pi}{k} a_2 Z_{(2,1)}^k(t) + \frac{\pi}{k} a_3 Z_{(2,2)}^k(t) + \dots + \frac{\pi}{k} a_n Z_{(2,n-1)}^k(t) = g(t^k). \quad (26)$$

To determine the unknown coefficients $a_j, j = 1, 2, 3, \dots, n$, we use the classical Galerkin method. To get n unknown and the same number of equations we multiply both sides of Eq. (26) by $w_{(2,k)}(t) Z_{(2,j)}^k(t), j = 0, 1, 2, \dots, n-1$ then integrating over the interval $[-1, 1]$ as well as use the results of orthogonality conditions Lemma 2.4 (20) to yield

$$a_j = \frac{2k^2}{\pi^2} \langle g, Z_{(2,j-1)}^k \rangle_{w_{(2,k)}(x)}, \quad j = 1, 2, \dots, n. \quad (27)$$

Case (II): The solution is bounded at both end-points $x = \pm 1$. For the bounded solution of Eq. (21), we impose the following condition

$$\int_{-1}^1 w_{(1,k)}(x) g(x^k) dx = 0, \quad (28)$$

In case 2, the unknown function $f(t)$ in (21) is approximated by

$$f_n(t) = b_0 w_{(2,k)}(t) Z_{(2,0)}^k(t) + w_{(2,k)}(t) \sum_{j=1}^n b_j Z_{(2,j)}^k(t), \quad k = 1, 3, \dots \quad (29)$$

Substitute Eq. (29) into Eq. (21) we have

$$\sum_{j=0}^n \int_{-1}^1 \frac{w_{(2,k)}(x)}{x^k - t^k} b_j Z_{(2,j)}^k(x) dx = g(t^k). \quad (30)$$

Due to Lemma 2.4 (19), we obtain

$$\frac{\pi}{k} b_0 Z_{(1,0)}^k(t) + \frac{\pi}{k} b_1 Z_{(1,1)}^k(t) + \frac{\pi}{k} b_2 Z_{(1,2)}^k(t) + \dots + \frac{\pi}{k} b_n Z_{(1,n)}^k(t) = g(t^k). \quad (31)$$

Proceeding in the same way and applying the orthogonality property, we get

$$b_m = -\frac{2k^2}{\pi^2} \int_{-1}^1 w_{(1,k)}(x) g(t^k) Z_{(2,m)}^k(x) dx = -\frac{2k^2}{\pi^2} \langle g, Z_{(1,m+1)}^k \rangle_{w_{(1,k)}}, \quad m = 0, 1, 2, \dots, n. \quad (32)$$

Most of the cases, the unknown coefficients in Eq. (32) are calculated exactly if it is not then Gauss-Chebyshev quadrature can be applied. We use Matlab to determine unknown coefficients $b_m, m = 0, 1, 2, 3, \dots, n$.

4. Numerical Examples

This section reports on two numerical experiments to study the accuracy and the performance of the proposed adaptive strategy method. All experiments have been conducted using MATLAB. Two examples were performed with $n = \{5, 7, 9\}$. We restrict the presentation to the simplest case $k = 3$.

Example 1 2 Consider the singular integral equation SSIE₃:

$$\int_{-1}^1 \frac{f(x)}{x^3 - t^3} dx = 8t^{12} - 8t^6 + 1, \quad t \in (-1, 1). \quad (33)$$

Case (I): Solution is unbounded at both end-points $x = \pm 1$. The exact solution of Eq. (33) is

$$f(t) = \frac{3t^2}{\pi\sqrt{1-t^6}}(8t^{15} - 12t^9 + 4t^3). \quad (34)$$

The approximate solution of Eq. (33) has a form

$$f_5(t) = a_0 Z_{(1,0)}^3(t) \frac{t^2}{\sqrt{1-t^6}} + \frac{t^2}{\sqrt{1-t^6}} \sum_{i=1}^5 a_i Z_{(1,i)}^3(t). \quad (35)$$

The unknown coefficients are found as follows

$$\begin{aligned} a_0 &= 0, & a_1 &= 0, & a_3 &= 0, \\ a_4 &= -0.47746482927568602599735201099929, \\ a_5 &= 0.47746482927568602599735201099929 \end{aligned}$$

The absolute errors between exact and approximation solutions are summarized in Table 2. The absolute errors between exact and approximation solutions are summarized in Table 2 and it reveals that proposed method is every accurate.

Case (II): Solution is bounded at both end-points $x = \pm 1$: The exact solution of Eq. (33) is:

$$f(t) = \frac{3}{\pi} t^2 \sqrt{1-t^6} (8t^9 - 4t^3). \quad (36)$$

The approximation solution of Eq. (33) has a form:

$$\phi_5(t) = t^2 \sqrt{1-t^6} a_0 Z_{(2,0)}^3(t) + t^2 \sqrt{1-t^6} \sum_{i=1}^5 a_i Z_{(2,i)}^3(t). \quad (37)$$

The unknown coefficients are found

$$\begin{aligned} b_0 &= 0, & b_1 &= 0, & b_3 &= 0, & b_4 &= 0, \\ b_4 &= 0.95492965855137199610918857921504, & b_5 &= 0, \end{aligned}$$

Absolute error are summarized in Table 3.

5. Conclusion

In this paper, the definition of the ECP of the first and second kinds of order k have been proposed and it is found that ECP are orthogonal with respect to weight functions $w_{(1,k)}(x) = \frac{x^{k-1}}{\sqrt{1-x^{2k}}}$, $w_{(2,k)}(x) = x^{k-1}\sqrt{1-x^{2k}}$ respectively on the interval $[-1, 1]$, where k is positive odd integer. Spectral properties of ECP of the first and second kinds $Z_{(i,n)}^k(x)$, $i = \{1, 2\}$ are proved. The density functions are characterized in terms of truncated series of ECPs of the first and second kinds. Table 2 and 3 reveal that proposed method is exact for the solution of the product of the form $f(t) = w_i(x)P_n(x)$.

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Table 1. Extended Chebyshev polynomials of the first and second kinds

n	$Z_{(1,n)}^k(x) = T_n(x^k)$	$Z_{(2,n)}^k(x) = U_n(x^k)$
0	1	1
1	x^k	$2x^k$
2	$2x^{2k} - 1$	$4x^{2k} - 1$
3	$4x^{3k} - 3x^k$	$8x^{3k} - 4x^k$
4	$8x^{4k} - 8x^{2k} + 1$	$16x^{4k} - 12x^{2k} + 1$
5	$x^k(16x^{4k} - 20x^{2k} + 5)$	$32x^{5k} - 32x^{3k} + 6x^k$
6	$32x^{6k} - 48x^{4k} + 18x^{2k} - 1$	$64x^{6k} - 80x^{4k} + 24x^{2k} - 1$

Table 2. Example 2 Case (I): Absolute Error

x	$n = 5$	$n = 7$	$n = 9$
0.9	5.828670879282072 E-16	5.828670879282072 E-16	5.828670879282072 E-16
0.7	1.110223024625157E-16	1.110223024625157E-16	1.110223024625157E-16
0.5	1.249000902703301 E-16	1.249000902703301 E-16	1.249000902703301 E-16
0.3	2.081668171172169 E-17	2.081668171172169 E-17	2.081668171172169 E-17
0.1	4.208059681959364e E-18	4.208059681959364e E-18	4.208059681959364e E-18
0.0	0.0000000000000000 E+00	0.0000000000000000 E+00	0.0000000000000000 E+00
-0.1	4.065758146820642 E-19	4.065758146820642 E-19	4.065758146820642 E-19
-0.3	2.081668171172169 E-17	2.081668171172169 E-17	2.081668171172169 E-17
-0.5	1.110223024625157-16	1.110223024625157-16	1.110223024625157-16
-0.7	5.551115123125783 E-17	5.551115123125783 E-17	5.551115123125783 E-17
-0.9	4.163336342344337 E-16	4.163336342344337 E-16	4.163336342344337 E-16

Table 3. Example 2 Case (II): Absolute Error

x	$n = 5$	$n = 7$	$n = 9$
0.9	3.053113317719181 E-16	3.191891195797325 E-16	3.191891195797325 E-16
0.7	5.551115123125783 E-17	5.551115123125783 E-17	5.551115123125783 E-17
0.5	8.326672684688674 E-17	8.326672684688674 E-17	8.326672684688674 E-17
0.3	1.908195823574488 E-17	1.908195823574488 E-17	1.908195823574488 E-17
0.1	4.214835945537399 E-18	4.214835945537399 E-18	4.214835945537399 E-18
0.0	0.0000000000000000 E+00	0.0000000000000000 E+00	0.0000000000000000 E+00
-0.1	4.133520782600986 E-19	4.336808689942018 E-19	4.336808689942018 E-19
-0.3	2.428612866367530 E-17	2.428612866367530 E-17	2.428612866367530 E-17
-0.5	1.387778780781446E-17	1.387778780781446E-17	1.387778780781446E-17
-0.7	5.551115123125783 E-17	5.551115123125783 E-17	5.551115123125783 E-17
-0.9	7.771561172376096 E-16	7.771561172376096 E-16	7.771561172376096 E-16