Mathematics and Statistics 10(3): 529-541, 2022 http://www.hrpub.org
DOI: 10.13189/ms.2022.100308

Three-Point Block Algorithm for Approximating Duffing Type Differential Equations

Ahmad Fadly Nurullah Rasedee[1,*], **Mohammad Hasan Abdul Sathar**[2],**Najwa Najib**[1], **Nurhidaya Mohamad Jan**[3], **Siti Munirah Mohd**[3], **Siti Nor Aini Mohd Aslam**[4]

[1]Faculty of Economics and Muamalat, Universiti Sains Islam Malaysia, Malaysia
[2]Centre of Foundation Studies for Agricultural Science, Putra University of Malaysia, Malaysia
[3]Kolej Genius Insan, Universiti Sains Islam Malaysia, Malaysia
[4]Mathematics Department, Kampus Dengkil, Universiti Teknologi Mara, Malaysia

Received November 30, 2021; Revised March 25, 2022; Accepted April 15, 2022

Cite This Paper in the following Citation Styles

***(a)**: [1] Ahmad Fadly Nurullah Rasedee, Mohammad Hasan Abdul Sathar, Najwa Najib, Nurhidaya Mohamad Jan, Siti Munirah Mohd, Siti Nor Aini Mohd Aslam, "Three-Point Block Algorithm for Approximating Duffing Type Differential Equations," Mathematics and Statistics, Vol.10, No.3, pp. 529-541, 2022. DOI: 10.13189/ms.2022.100308*

***(b)**: Ahmad Fadly Nurullah Rasedee, Mohammad Hasan Abdul Sathar, Najwa Najib, Nurhidaya Mohamad Jan, Siti Munirah Mohd, Siti Nor Aini Mohd Aslam, (2022). Three-Point Block Algorithm for Approximating Duffing Type Differential Equations. Mathematics and Statistics, 10(3), 529-541, DOI: 10.13189/ms.2022.100308*

Abstract The current study was conducted to establish a new numerical method for solving Duffing type differential equations. Duffing type differential equations are often linked to damping issues in physical systems, which can be found in control process problems. The proposed method is developed using a three-point block method in backward difference form, which offers an accurate approximation of Duffing type differential equations with less computational cost. Applying an Adam's like predictor-corrector formulation, the three point block method is programmed with a recursive relationship between explicit and implicit coefficients to reduce computational cost. By establishing this recursive relationship, we established a corrector algorithm in terms of the predictor. This eliminates any undesired redundancy in the calculation when obtaining the corrector. The proposed method allows a more efficient solution without any significant loss of accuracy. Four types of Duffing differential equations are selected to test the viability of the method. Numerical results will shows efficiency of the three-point block method compared against conventional and more established methods. The outcome of this research is a new method for successfully solving Duffing type differential equation and other ordinary differential equations that are found in the field of science and engineering. An added advantage of the three-point block method is its adaptability to parallel programming.

Keywords ODEs, Block, Multistep Method, Variable Order Step Size

---

1 Introduction

Differential equations are widely used to model real-life application, ranging from the simplest financial problems to the most complex engineering problems. These problem are often modelled in various form of differential equations. For the current research, let's consider the general form of the initial value higher order ordinary differential equation (ODE), $y^{(d)}$ which is denoted as

$$y^{(d)} = f(t, \hat{Y}) \tag{1.1}$$

where

$$\hat{Y} = (y, y', y'', \dots, y^{(d-1)})$$

and

$$\hat{\xi} = (\xi, \xi', \xi'', \dots, \xi^{(d-1)})$$

fulfil the initial conditions $\hat{Y}(a) = \hat{\xi}$ in the given interval $a \le t \le b$. Higher order ODEs can be found in many form but the most interesting problems are commonly in the form of non-linear problems with oscillatory solutions. For such reason, authors decided for the current research to focus specifically on the Duffing differential equation.

The Duffing differential equation was originally founded by Duffing [1], which was a non linear second order differential equation. It was derived in efforts to model certain damped and driven oscillators. The Duffing equation in its current form is denoted as

$$y''(t) + \delta y' + \alpha y + \beta y^3 = \gamma \cos(\omega t)$$

with the parameters δ, α, β, γ and ω. These parameters are given constants where, δ controls the damping amount, α controls the linear stiffness and where the non linearity is dictated by β. Next, the parameter γ is used to denote amplitude of a periodic driving force and ω as its corresponding angular frequency. The application of Duffing oscillators can be found in problems related to electronic circuits. We refer the reader to latest works involving the application of Duffing type oscillators which are examined by [2, 3, 4, 5, 6, 7].

As many other non-linear differential equations, very so often analytical solution are not attainable. In such cases, numerical solution are required. Among researcher that have studied numerical solutions particularly for the Duffing differential equation include the more classical works by authors such as [8, 9, 10, 11] whereas recent research on the subject includes [12, 13, 14, 15, 16] and many more.

In this article, authors proposed a block multistep method for obtaining numerical approximation of Duffing differential equation. The proposed method is selected because of the elegance of the block multistep method when formulated in backward difference form which has the ability to be adopted with a parallel programming if required. The current works apply a variable order constant step three point block method (up to order 6) which approximate three points of equidistant simultaneously or denoted as the 3PBBD method. The 3PBBD method follows a series of research which was inspired by authors such as [17, 18, 19, 20, 21].

Numerical codes for solving ODEs were considered robust and would have been overlooked if it was not for Krogh [18]. Since then, works as [19] and [20] kindled interest for producing new numerical codes for solving ODEs. In contrast the divided difference formulation developed in [21], the method proposed here implements an Adams like predictor-corrector algorithm in three point block mode with a backward difference formulation. This method is able to reduce computational cost (time steps) significantly. Numerical results will be compared with other established numerical method and run using C algorithm. Results will show the advantage of the 3PBBD method in terms of totals steps and calculation time with insignificant reduction of accuracy, if any. For the latest numerical multistep based method, readers can refer to works by [23, 24, 25, 26, 27, 28, 29, 30, 31].

2 Derivation of Three Point Block Multistep Method

The current section will provide the derivation of the third point. For the the first and second point derivation, the reader can refer to [32] and [33] respectively.

2.1 Explicit Integration Coefficients

Consider the higher order ODE, $y^{(d)}$ in (1.1). We begin with the derivation of the explicit coefficient by integrating (1.1) once,

$$y^{(d-1)}(t_{n+3}) = y^{(d-1)}(t_n) + \int_{t_n}^{t_{n+3}} f(t, \hat{Y})\, dt \tag{2.2}$$

Next, by implementing the following Newton-Gregory polynomial

$$P_n(t) = \sum_{i=0}^{k-1} (-1)^i \begin{pmatrix} -s \\ i \end{pmatrix} \nabla^i f_n ds,$$

where $s = \frac{t-t_n}{h}$ and $dt = hds$ changes the limit of integration, hence giving

$$y^{(d-1)}(t_{n+3}) = y^{(d-1)}(t_n) + \int_0^3 \sum_{i=0}^{k-1} (-1)^i \left(\begin{array}{c} -s \\ i \end{array} \right) \nabla^i f_n ds.$$

By denoting

$$\gamma_{3,1,i} = (-1)^i \int_0^3 \left(\begin{array}{c} -s \\ i \end{array} \right) ds,$$

our estimation becomes

$$y^{(d-1)}(t_{n+3}) = y^{(d-1)}(t_n) + h \sum_{i=0}^{k-1} \gamma_{3,1,i} f_n ds.$$

Subsequently,

$$G_{3,1}(t) = \sum_{i=0}^{\infty} \gamma_{3,1,i} t^i$$

is defined as the generating function for the set of coefficients $\gamma_{3,1,i}$. Through mathematical induction where

$$G_{3,1}(t) = \int_0^3 e^{-s \log(1-t)} dt$$

the generating function (GF) can be estimated as follows

$$G_{3,1}(t) = -\left[\frac{(1-t)^{-3} - 1}{\log(1-t)} \right].$$

Next is the derivation for the second order predictor. Again, we consider Eq (1.1) but with the difference of integrating it twice as follows

$$\begin{aligned} y^{(d-2)}(t_{n+3}) =& y^{(d-2)}(t_n) + hy^{(d-1)}(t_n) \\ &+ h \int_{t_n}^{t_{n+3}} (t_{n+3} - t) f\left(t, \hat{Y}\right) dt. \end{aligned} \tag{2.3}$$

Substituting $f\left(t, \hat{Y}\right)$ once more with the Newton Gregory backward difference polynomial but this time replacing $(t - t_n)$ with $h(3 - s)$ where s as previously defined, yields

$$\begin{aligned} y^{(d-2)}(t_{n+3}) &= y^{(d-2)}(t_n) + hy^{(d-1)}(t_n) \\ &+ h^2 \int_0^3 (3-s) \sum_{i=0}^{k-1} (-1)^i \binom{-s}{i} \nabla^i f_n \, ds. \end{aligned} \tag{2.4}$$

The integral above is then replaced with

$$\gamma_{3,2,i} = (-1)^i \int_0^3 (3-s) \binom{-s}{i} ds, \tag{2.5}$$

establishing the following second order predictor formula

$$\begin{aligned} y^{(d-2)}(t_{n+3}) =& y^{(d-2)}(t_n) + hy^{(d-1)}(t_n) \\ &+ h^2 \sum_{i=0}^{k-1} \gamma_{3,2,i} t^i \nabla^i f_n. \end{aligned} \tag{2.6}$$

By solving the integral, we can establish second order GF for the set of coefficients $\gamma_{3,2,k}$ as

$$G_{3,2}(t) = \left[\frac{3}{\log(1-t)} - \frac{G_{3,1}(t)}{\log(1-t)} \right]$$

Applying similar steps as the previous orders with some slight alterations, the d-th order generating function can be estimated as

$$G_{3,(d)}(t) = \frac{1}{(d-1)!} \left[\frac{3^{(d-1)}}{\log(1-t)} - \frac{(d-1)! G_{3,(d-1)}(t)}{\log(1-t)} \right].$$

By substituting

$$G_{3,2}(t) = \sum_{i=0}^{k} \gamma_{3,2,i}$$

and applying expansion techniques general formulation for explicit integration coefficient have the following relationship

$$\gamma_{3,(d),0} = \gamma_{3,(d-1),1}$$

$$\gamma_{3,(d),k} = \gamma_{3,(d-1),k+1} - \sum_{i=0}^{k-1} \left(\frac{\gamma_{3,(d),i}}{k-i+1} \right) \quad k = 1, 2, \ldots.$$

2.2 Implicit Integration Coefficients

Next, proceed with the implicit integration coefficient. The higher order ODE (1.1), is approximated again by (2.2). Once more, by Newton-Gregory's polynomial but with a slight difference of changing $s = \frac{t-t_{n+3}}{h}$ a new limit of integration is obtained, where

$$y^{(d-1)}(t_{n+3}) = y^{(d-1)}(t_n) + \int_{-3}^{0} \sum_{i=0}^{k-1} (-1)^i \left(\begin{array}{c} -s \\ i \end{array} \right) \nabla^i f_{n+3} ds,$$

Now, $\gamma^*_{3,1,i}$ is denoted by

$$\gamma^*_{3,1,i} = (-1)^i \int_{-3}^{0} \left(\begin{array}{c} -s \\ i \end{array} \right) ds.$$

which provides the corrected estimation of $y^{(d-1)}(t_{n+3})$ as follows

$$y^{(d-1)}(t_{n+3}) = y^{(d-1)}(t_n) + h\sum_{i=0}^{k-1}\gamma^*_{3,1,i} f_{n+3} ds.$$

Then, the set of coefficients $\gamma^*_{3,1,i}$ are defined in form of the GF, $G^*_{3,1}(t)$ as

$$G^*_{3,1}(t) = \sum_{i=0}^{\infty}\gamma^*_{3,1,i}.$$

Again, by denoting

$$G^*_{3,1}(t) = \int_{-3}^{0} e^{-s\log(1-t)}dt$$

and followed by mathematical induction, the implicit GF can be estimated by

$$G^*_{3,1}(t) = -\left[\frac{1}{\log(1-t)} - \frac{(1-t)^3}{\log(1-t)}\right].$$

Due to the similar nature of the derivation techniques, we forward to the n^{th} order derivation. Once again, we integrate (1.1). For the d^{th} order corrector, (1.1) is integrated d folds, thus establishing the following

$$\begin{aligned} y(t_{n+3}) =& y(t_n) + hy'(t_n) + \ldots + \frac{h^{(d-1)}}{(d-1)!}y^{(d-1)}(t_n) \\ &+ h\int_{t_n}^{t_{n+3}} \frac{(t_{n+3}-t)^{(d-1)}}{(d-1)!} f\left(t, \hat{Y}\right)\, dt, \end{aligned} \tag{2.7}$$

thus yielding

$$\begin{aligned} y(t_{n+3}) = y(t_n) + hy'(t_n) + \ldots + \frac{h^{(d-1)}}{(d-1)!}y^{(d-1)}(t_n) \\ + h^{(d-1)}\sum_{i=0}^{k-1}\beta_{3,n,i}\nabla^i f_{n+3}. \end{aligned} \tag{2.8}$$

The generalized implicit GF can be mathematically deduced, thus presenting the following formula

$$\begin{aligned} G^*_{3,(d)}(t) = \frac{1}{(d-1)!}\times \\ \times\left[\frac{3^{(d-1)}(1-t)^3}{\log(1-t)} - \frac{(d-1)!G^*_{3,(d-1)}(t)}{\log(1-t)}\right]. \end{aligned}$$

with the corresponding implicit integration coefficients of different orders written in form of the following recursive relationship

$$\begin{aligned} \gamma^*_{3,(d),0} &= \gamma^*_{3,(d-1),1} \\ \gamma^*_{3,(d),k} &= \gamma^*_{3,(d-1),k+1} - \sum_{i=0}^{k-1}\left(\frac{\gamma^*_{3,(d),i}}{k-i+1}\right) \\ k &= 1, 2, \ldots \end{aligned}$$

To increase computational efficiency it's essential to reduce certain redundancy hence, the need to obtain a recurring relationship between explicit and implicit coefficients and coefficients of different orders is established. Similar to the relationship between integration coefficient provided in [32] and [33], we obtain the following relationship

$$\sum_{i=0}^{\infty}\gamma^*_{3,j,i}t^i = (1-t)^3\sum_{i=0}^{\infty}\gamma_{3,j,i}t^i.$$

A few examples of explicit and implicit integration coefficients that were generated are presented in Table 1 and 2 below.

Table 1. Explicit integration coefficients.

i	0	1	2	3	4	5	6
$\gamma_{3,1,i}$	3	$\frac{9}{2}$	$\frac{27}{4}$	$\frac{75}{8}$	$\frac{987}{80}$	$\frac{2499}{160}$	$\frac{43021}{2240}$
$\gamma_{3,2,i}$	$\frac{9}{2}$	$\frac{9}{2}$	$\frac{45}{8}$	$\frac{552}{80}$	$\frac{1323}{160}$	$\frac{21762}{2240}$	$\frac{50319}{4480}$

Because the criteria for accepting an order change is instrumental for variable order in a method (refer to [14]), the current research considers the error estimate E_n as a parameter for determining order increment or reduction.

Order reduction

Case 1: If $n > 2$ and $\max(|E_{(n-1)}|, |E_{(n-2)}|) \leq |E_n|$.

Case 2: If $E_{(n+1)}$ exist,for $n > 1$ if$|E_{(n-1)}| \leq \min(|E_n|, |E_{(n+1)}|)$.

Order increment

Order will increase only if $n+1$ successful step have been achieved.

Case 1: If for $n > 1$, $|E_{(n+1)}| < |E_n| < \max(|E_{(n-1)}|, |E_{(n-2)}|)$

Case 2: If $n = 1$, $|E_{(n+1)}| < 0.5|E_n|$.

The order of the method for the current research is restricted to the range $1 \leq n \leq 6$.

Table 2. Implicit integration coefficients.

i	0	1	2	3	4	5	6
$\gamma_{3,1,i}$	3	$-\frac{9}{2}$	$\frac{9}{4}$	$-\frac{3}{8}$	$-\frac{3}{80}$	$-\frac{3}{160}$	$-\frac{29}{2240}$
$\gamma_{3,2,i}$	$-\frac{9}{2}$	$\frac{9}{2}$	$-\frac{9}{8}$	$\frac{3}{20}$	$\frac{9}{160}$	$\frac{9}{224}$	$\frac{141}{4480}$

3 Order, Stability and Convergence

The current section discusses conditions for order and zero stability of block methods similarly as discussed by authors such as [34, 35, 31] and etc. Order and stability presented here are for the three-point block algorithm which follows estimation discussed in [35]. Here are preliminary definitions that are required.

3.1 Preliminaries

Definitions and criteria required to show stability, prove convergence and also determine order of the method are provided in the current subsection. Firstly, we will present definitions that are crucial for a linear multistep method (LLM).

Definition 3.1 *Let denote the general linear multistep method by*

$$\sum_{i=0}^{k}\phi_i p_{n+i} = h\sum_{i=0}^{k}\psi_j q_{n+i}$$

Next, the linear differential operator for the general LLM is defined as follows

Definition 3.2 *The linear differential operator ℓ which corresponds with the LLM is defined by*

$$\ell[q(t);h] := \sum_{i=0}^{k}[\phi_i q(t+ih) - h\psi_i q'(t+ih)],$$

where $q(t)$ is an arbitrary function in $C^1[a,b]$.

Next, consider the function $q(t)$ to be d times differentiable. Then, the function $q(t+ih)$ and and its derivative $q'(t+ih)$ is expanded with respect to t and rearranged as

$$\ell[q(t);h] := C_0 q(t) + C_1 h q'(t) + \ldots + C_d h^d q^{(d)}(t).$$

Note that the set of coefficients C_i, $i = 0, 1.\ldots, d, \ldots$ are constants.

Definition 3.3 *The LMM thats corresponds to the difference operator ℓ are said to be of order d if, $C_0 = C_1 = \ldots = C_d = 0, \quad C_{d+1} \neq 0$ where*

$$\begin{aligned}
C_0 =&\phi_0 + \phi_1 + \ldots + \phi_k\\
C_1 =&(\phi_1 + 2\phi_2 + \ldots + k\phi_k) - (\psi_1 + \psi_2 + \ldots + \psi_k)\\
\vdots\quad & \vdots\\
C_b =&\frac{1}{b}(\phi_1 + 2\phi_2^b + \ldots + k^b\phi_k)\\
&- \frac{1}{d-1}(\psi_1 + 2^{b-1}\psi_2 + \ldots + k^{b-1}\psi_k)
\end{aligned}$$

Definition 3.4 *The zero stability of a block method is zero stable if the roots r_i of the first characteristic polynomial $\rho(r)$ denoted by*

$$\rho(r) = \det\left(\sum_{i=0}^{m} A_i r^{m-i}\right) = 0, \quad i = 1, \ldots m$$

fulfil the requirements $|r_i| \leq 1$ given the roots $|r_i| = 1$, where the multiplicity ≤ 2.

Definition 3.5 *A LLM is said to be consistent if it is of the order $p \geq 1$.*

3.2 Order of the method

Predictor and corrector in the current section is predetermined with $k = 6$ number of back values. Consider the example f', by way of definition 3.1 and 3.2, predictor and corrector in the form of y_{i+b} of the three block methods can be expressed as

$$\begin{aligned}
y_{i+1} = y_i + h\bigg(&\frac{1901}{720}\gamma_i - \frac{1387}{360}\gamma_{i-1} + \frac{109}{30}\gamma_{i-2}\\
&-\frac{637}{360}\gamma_{i-3} + \frac{251}{720}\gamma_{i-4}\bigg)
\end{aligned} \tag{3.9}$$

$$y_{i+2} = y_i + h\left(\frac{1079}{90}\gamma_i - \frac{1198}{45}\gamma_{i-1} + \frac{424}{15}\gamma_{i-2} - \frac{658}{45}\gamma_{i-3} + \frac{269}{90}\gamma_{i-4}\right) \quad (3.10)$$

$$y_{i+3} = y_i + h\left(\frac{2877}{80}\gamma_i - \frac{3819}{40}\gamma_{i-1} + \frac{1089}{10}\gamma_{i-2} - \frac{2349}{40}\gamma_{i-3} + \frac{987}{80}\gamma_{i-4}\right) \quad (3.11)$$

and

$$y_{i+1} = y_i + h\left(\frac{95}{288}\gamma_{i+1} + \frac{1427}{1440}\gamma_i - \frac{133}{240}\gamma_{i-1} + \frac{241}{720}\gamma_{i-2} - \frac{173}{1440}\gamma_{i-3} + \frac{3}{160}\gamma_{i-4}\right) \quad (3.12)$$

$$y_{i+2} = y_i + h\left(\frac{14}{45}\gamma_{i+2} + \frac{43}{30}\gamma_{i+1} + \frac{7}{45}\gamma_i + \frac{14}{90}\gamma_{i-1} - \frac{6}{90}\gamma_{i-2} + \frac{1}{90}\gamma_{i-3}\right) \quad (3.13)$$

$$y_{i+3} = y_i + h\left(\frac{51}{160}\gamma_{i+3} + \frac{219}{160}\gamma_{i+2} + \frac{57}{80}\gamma_{i+1} + \frac{57}{80}\gamma_i - \frac{21}{160}\gamma_{i-1} + \frac{3}{160}\gamma_{i-2}\right) \quad (3.14)$$

respectively. Then rearrange the predictor and corrector in way of coefficients such that conditions in Definition 3.3 are satisfied, we obtain for the predictor;

$$C_0 = C_1 = \ldots = C_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad C_6 = \begin{pmatrix} \frac{95}{288} \\ \frac{33}{10} \\ \frac{2499}{160} \end{pmatrix}$$

and the corrector;

$$C_0 = C_1 = \ldots = C_6 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad C_7 = \begin{pmatrix} -\frac{863}{60480} \\ -\frac{37}{3780} \\ -\frac{99}{2240} \end{pmatrix}.$$

By Definition 3.3 the predictor is of order 5 and the corrector is of order 6.

3.3 Zero Stability

The viability of a linear multistep method is said to be governed by conditions of zero stability. Using similar conditions as presented in [20], zero stability of the three-point block method can be established by considering the standard linear test problem

$$y' = \lambda y.$$

Applying equations (3.9) to (3.11) and (3.12) to (3.14), to conditions affiliated with Definition 3.4 yields the stability polynomial of both predictor and corrector as

$$\rho_p(t;0) = \rho_c(t;0) = t^6 - t^5$$

with roots $t_0 = \ldots = t_5 = 0$, $t = 1$. Thus, Definition 3.4 dictates that both predictor and corrector formulae are zero stable for predictor of order 5 and corrector of order 6.

4 Convergence the backward difference method

For convergence of the three-point block method, we refer to the works of [38]. As discussed in Suleiman [38], sufficient conditions to prove convergence is governed by the following.

Theorem 4.1 *1 Conditions necessary for the LLM* (2.6) *and* (2.8) *to be convergent are*
i) The block method must be consistent
ii) The block method must be zero stable

Authors would like readers to refer to works of [38] for proof of the theorem.
Condition in Definition 3.5 validates that the predictor as cosistent of order 5 and corrector as consistent of order 6 respectively. And as shown in the previous subsection, both methods are zero stable thus satisfying the necessary conditions for convergence.

5 Numerical Approximation

The current research compare numerical results of the 3PBBD method against various multi step and conventional methods. Accuracy, total steps and calculation time are instruments used to validate the efficiency and effectiveness of the proposed 3PBBD method. Problems 1 and 2 are second order Duffing oscillator selected to compare the efficiency of the 3PBBD method against other multistep method. Whereas, Problem 3 is a Duffing equation without any exact solution to show the accuracy of its approximated solution. Problem 4 on the other hand, is a fourth order problem chosen for the level of difficulty which provides a challenging approximation for the 3PBBD method. Listed below are abbreviations that are used through out this section.

STEPS: total steps,
MAXE: the overall maximum error,
AVR: the average error,
MTD: the method used
TIME: Calculation time (in seconds)
DI: direct integration,
SNM: standard numerical
SHPM: standard homotopy perturbation
1PBBD: One-Point block method
2PBBD: Two-Point block method
3PBBD: Three-Point block method

5.1 Error Estimation

The errors $(e_i)_t$ computed in the research are governed by the following equation

$$(e_i)_j = \left|\frac{(y_i)_j - (y(t_i))_j}{A + B(y(t_i))_j}\right|.$$

The given $(y)_j$ is defined as the j^{th} component of y where, the errors $(e_i)_j$ corresponds to
If $A = 1, B = 0$ denotes the absolute error test
If $A = 1, B = 1$ denotes the mixed error test
If $A = 0, B = q$ denotes the relative error test
The maximum error is given by

$$MAXE = \max_{1<i<STEPS}\left(\max_{1<i<N}(e_i)_j\right)$$

whereas the average is defined

$$AVR = \left(\sum_{i=1}^{STEPS}\sum_{t=1}^{N}(e_i)_j\right)/(N)(STEPS)$$

with N representing the number of equations in the systems.

5.2 Test Problems

Problem 1: The equation $y''(t) + 2y'(t) + y(t) + 8y^3(t) = e^{-3t}$ for $0 \leq t \leq 100$ was obtained from [36] and the given IVC $y(0) = \frac{1}{2}$, $y'(0) = -\frac{1}{2}$ and $y(t) = \frac{1}{2}e^{-t}$ as the exact solution.

Problem 2: The equation $y''(t) + y(t) + y'(t) + y^2(t)y'(t) = 2\cos t - \cos^3 t$ for $0 \leq t \leq 100$ was obtained from [37] and the given IVC $y(0) = 0$, $y'(0) = 1$ and $y(t) = \sin t$ as the exact solution.

Problem 3: The equation $y''(t) + y(t) + y^3(t) = 0$ for $0 \leq t \leq 5$ was obtained from [**?**] and the given IVC $y(0) = 1$, $y'(0) = 0$ and without any known exact solution.

Problem 4: The equation $y''''(t) + 5y''(t) + 4y(t) - \frac{1}{6}y^3(t) = 0$ for $0 \leq t \leq 14$ was obtained from [8] and the given IVC $y(0) = 0$, $y'(0) = 1.91103$, $y''(0) = 0$, $y'''(0) = -1.15874$ and $y(t) = 2.1906\sin 0.9t - 0.02247\sin 2.7t + 0.000045\sin 4.5t$ as the exact solution.

Problem 5: The equation $y''(t) + 0.05y'(t) + y^3(t) = 7.5\cos t$ for $0 \leq t \leq 100$ artificial problem and the given IVC $y(0) = 0$, $y'(0) = 1$ and without any known exact solution.

5.3 Numerical Results and Discussion

For the current problems, a six order method (total of six back values) was implemented to approximate the solutions. Detailed explanation for order selection of the proposed method can be found in [40]. Numerical results presented in Table 3 compare the accuracy, total steps and calculation time of three multistep methods, 1PBBD [32], 2PBBD [33] and 3PBBD for Problems 1 and 2. As expected, some loss of accuracy might occur when implementing a block algorithm but its accuracy is still as competitive (within the same order for each step size). In terms of total step size and calculation cost (time), the 3PBBD method is shown to be superior compared to its counterparts.

Fig.1 to 4, show the efficiency of the 3PBBD method. As suggested in [40], the efficiency of the multistep method can be defined as the under-most curve of the graph. In order to measure the efficiency, the total steps and computational time are compared against the order of accuracy (step size). As illustrated in Fig. 1 - 4, it is evident that the 3PBBD method is proven to be the under-most curve of all three methods hence, the 3PBBD is the most efficient.

Approximated results of SNM and SHPM methods were obtained from [39]. Results presented in Table 4, show that the accuracy of the 3PBBD method rivals the approximated solution of other established meth-

Table 3. Comparison of total steps and accuracy for Problems 1 and 2.

h	MTD	Problem 1			Problem 2		
		STEPS	MAXE	TIME	STEPS	MAXE	TIME
10^{-1}	1PBBD	1000	$1.40393(-04)$	0.001	1000	$7.14382(-04)$	0.014
	2PBBD	500	$1.37005(-04)$	0.001	500	$2.00067(-03)$	0.014
	3PBBD	335	$1.45239(-03)$	0.001	335	$4.54964(-03)$	0.014
10^{-2}	1PBBD	10000	$8.73746(-07)$	0.013	10000	$6.99653(-06)$	0.015
	2PBBD	5000	$1.62099(-06)$	0.011	5000	$1.46079(-05)$	0.015
	3PBBD	3335	$4.50074(-06)$	0.011	3335	$3.18752(-05)$	0.015
10^{-3}	1PBBD	100000	$8.28441(-09)$	0.078	100000	$7.02200(-08)$	0.125
	2PBBD	50000	$1.64442(-08)$	0.076	50000	$1.41060(-07)$	0.109
	3PBBD	33335	$3.78240(-08)$	0.069	33335	$3.16289(-07)$	0.093
10^{-4}	1PBBD	1000000	$8.23794(-11)$	0.681	1000000	$7.51013(-10)$	0.921
	2PBBD	500000	$1.64681(-10)$	0.656	500000	$1.40203(-09)$	0.921
	3PBBD	333335	$3.71329(-10)$	0.546	333335	$3.16197(-09)$	0.906
10^{-5}	1PBBD	10000000	$1.15053(-12)$	8.858	10000000	$5.31186(-10)$	9.310
	2PBBD	5000000	$1.90458(-12)$	6.552	5000000	$5.10937(-10)$	8.921
	3PBBD	3333335	$3.72584(-12)$	5.217	3333335	$2.28066(-10)$	8.198

Table 4. Numerical result for Problem 3.

t	3PBBD			SHPM	SNM
	$h = 1 \times 10^{-1}$	$h = 1 \times 10^{-3}$	$h = 1 \times 10^{-5}$		
0.5	7.71225(-1)	7.68802(-1)	7.68802(-1)	7.68766(-1)	7.68802(-1)
1.0	2.39146(-1)	2.33692(-1)	2.33692(-1)	2.33680(-1)	2.33692(-1)
2.0	-8.55526(-1)	-8.59349(-1)	-8.59349(-1)	-8.9323(-1)	-8.59349(-1)
3.5	-1.08744(-1)	-9.30123(-2)	-9.30110(-2)	-9.30340(-2)	-9.30130(-2)
5.0	9.52501(-1)	9.47130(-1)	9.47130(-1)	9.47107(-1)	9.47130(-1)

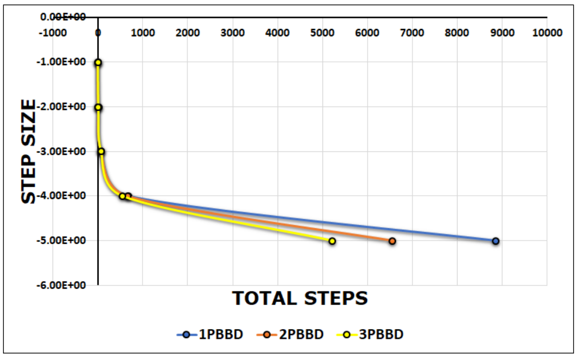

Figure 1. Total steps over accuracy for Problem 1.

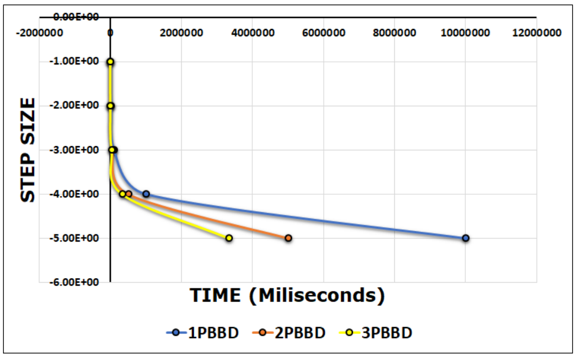

Figure 2. Computational time over accuracy for Problem 1.

ods (SNM and SHPM), especially when using smaller step sizes. Whereas, Fig. 5 and 6 compare the 3PBBD method with the preset Euler method in Mathematica 12. Graphs plotted in both figures over-line on top of each other signifying the similarity in accuracy of both methods. Fig. 7 is included for a more visual comprehension of the approximated values by 3PBBD of the Problem 3.

Table 5 consists of numerical results for Problem 4, which was intended to test the proposed method with a problem of high difficulty. In Table 5, numerical results of the 3PBBD method are compared against DI, 1PBBD and 2PBBD methods. The DI method is a multistep method established by Suleiman [22] which is based on a divided difference formulation. The DI was originally established as a viable multistep method for solv-

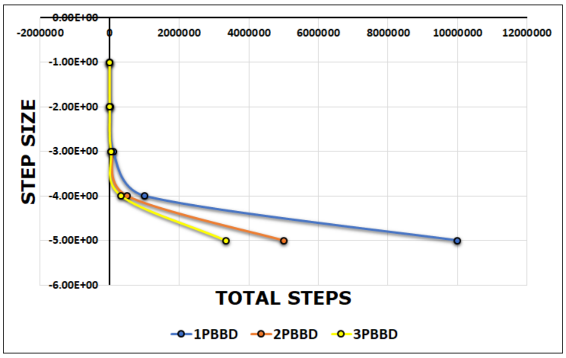

Figure 3. Total steps over accuracy for Problem 2.

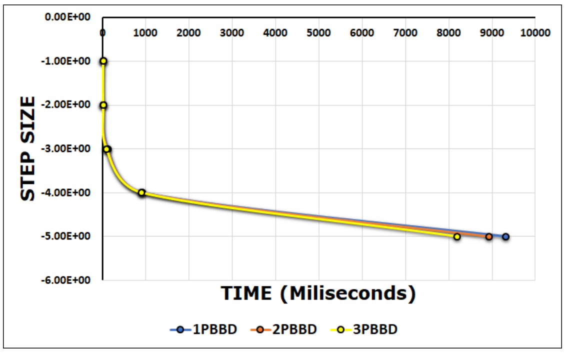

Figure 4. Computational time over accuracy for Problem 2.

ing higher order ODEs but, in the current research it has been restricted to a constant step algorithm to provide a fair comparison between both methods. In the current table, the DI is used as a basis of accuracy to validate accuracy of the 3PBBD method. As shown in Table 5, with the exception of step size $h = 10^{-1}$ the 3PBBD is able to match the accuracy of the DI method. When a smaller step size is selected, the accuracy of 3PBBD converges to a similar approximation as the DI.

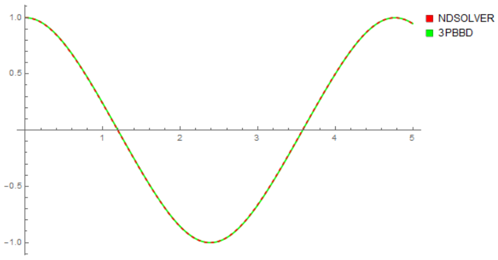

Figure 5. Comparison of solution between 3PBBD and NDSOLVER for Problem 3.

Finally, Fig. 8-9 show the approximated solutions obtained by the 3PBBD for solving Problem 5. Problem 5 is an artificial duffing equation chosen for its dynamic nature. This problems provides a higher level of difficulty and will further validate the capability of the 3PBBD. Fig. 8 illustrates a point to point parametric image between y and y'. Finally, Fig. 9 shows a three dimensional parametric y and y' in respect to each corresponding step t. The figure clearly shows dynamics behaviour of Problem 5 subsequently highlighting its level of difficulty.

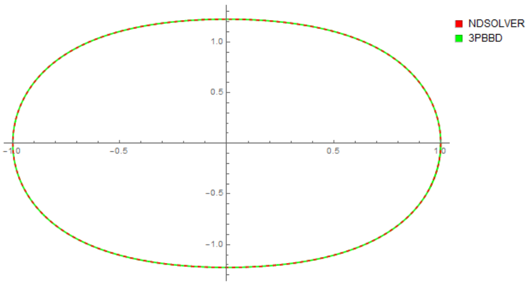

Figure 6. Comparison of parametric solution between 3PBBD and NDSOLVER for Problem 3.

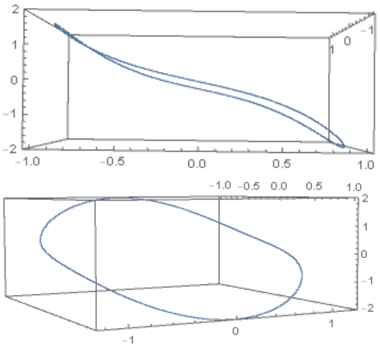

Figure 7. 3D parametric representation of Problem 3.

6 Conclusions

The proposed 3PBBD method is a viable alternative for solving higher order Duffing oscillator. The 3PBBD method requires less computational steps, which translate to less computational cost (time) without significant loss of accuracy. Even though the current 3PBBD methods is coded sequentially, it offers less computational time than its multistep counterpart. The added advantage that the 3PBBD provides is its ability to be programmed with parallel coding which will reduce computational cost even more significantly. The lack of performance at certain step size is contributed to stability issues involving the order of the method.

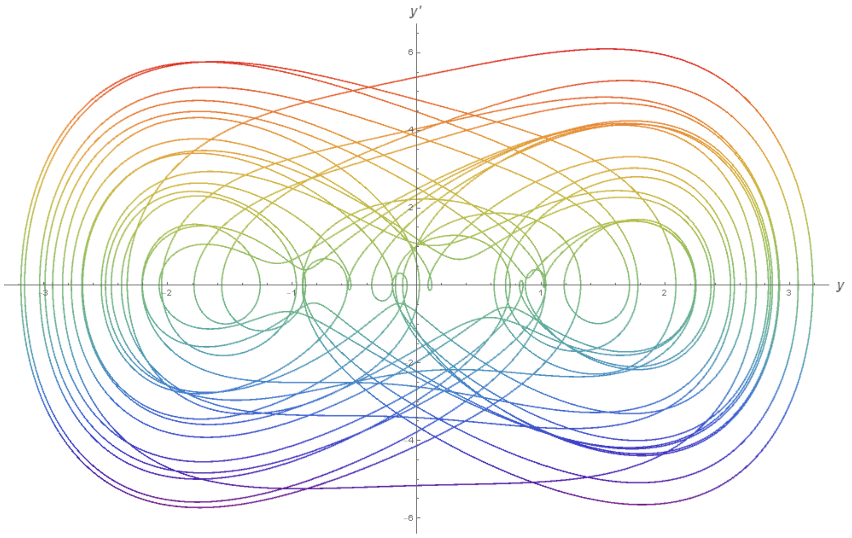

Figure 8. A parametric solution of y and y' for Problem 5.

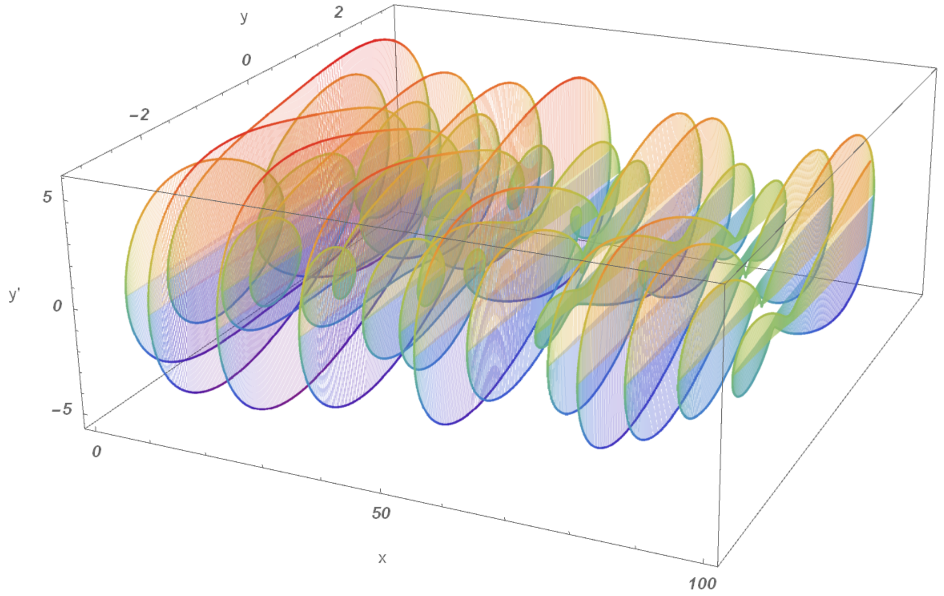

Figure 9. 3D parametric representation of Problem 5.

Table 5. Comparison of total steps and accuracy for Problem 4.

h	MTD	STEPS	MAXE	AVR
10^{-1}	DI	50	7.98200(−05)	2.73779(−05)
	1PBBD	50	8.24861(−04)	2.91423(−04)
	2PBBD	25	1.82220(−03)	1.82220(−03)
	3PBBD	17	3.51926(−03)	1.42600(−03)
10^{-2}	DI	500	6.58731(−05)	3.16421(−05)
	1PBBD	500	6.98992(−05)	3.11129(−05)
	2PBBD	250	7.73378(−05)	3.30744(−05)
	3PBBD	167	9.67353(−05)	3.10846(−05)
10^{-3}	DI	5000	6.58826(−05)	3.16867(−05)
	1PBBD	5000	6.58253(−05)	3.16804(−05)
	2PBBD	2500	6.59973(−05)	3.16795(−05)
	3PBBD	1667	6.56246(−05)	3.16535(−05)
10^{-4}	DI	50000	6.58826(−05)	3.16861(−05)
	1PBBD	50000	6.58821(−05)	3.16860(−05
	2PBBD	25000	6.58838(−05)	3.16862(−05)
	3PBBD	16667	6.58801(−05)	3.16853(−05)
10^{-5}	DI	500000	6.58826(−05)	3.16860(−05)
	1PBBD	500000	6.58826(−05)	3.16860(−05)
	2PBBD	250000	6.58826(−05)	3.16860(−05)
	3PBBD	166667	6.58826(−05)	3.16860(−05)

Acknowledgements

This research has been supported by Universiti Sains Islam Malaysia and the Ministry of Higher Education (MoHE), Malaysia under the Fundamental Research Grant Scheme (FRGS), project number USIM/FRGS/FEM/055002/51517.

REFERENCES

[1] G. Duffing. Erzwungene Schwingungen bei ver¨anderlicher Eigenfrequenz und ihre technische Bedeutung, F. Vieweg & sohn, Braunschweig, Germany, 1918.

[2] K. Rajagopal, J. R. M. Pone, S. T. Kingni, S. T., Arun, S. and A. Karthikeyan. Analysis and electronic implementation of an absolute memristor autonomous Van der Pol-Dffing circuit, The European Physical Journal Special Topics, Vol. 228, No. 10, 2287–2299, 2019.

[3] L. Kengne, J. R. M. Pone, H. T. K Tagne and J. Kangne. Dynamics, control and symmetry breaking aspects of a modified van der Pol–Duffing oscillator, and its analog circuit implementation, Analog Integrated Circuits and Signal Processing, Vol. 103, No. 1, 73-93, 2020.

[4] W. Jiang, H. Shi, X. Han, L. Chen and Q. Bi. Double jump broadband energy harvesting in a Helmholtz–Duffing Oscillator, Journal of Vibration Engineering & Technologies, Vol. 8, No. 6, 893-908, 2020.

[5] Y. Jia. Review of nonlinear vibration energy harvesting: Duffing, bistability, parametric, stochastic and others, Journal of Intelligent Material Systems and Structures, Vol. 31, No. 7, 921-944, 2020.

[6] M. G. Clerc, S. Coulibaly and M. A. Ferr´e. Freak chimera states in a locally coupled Duffing oscillators chain, Communications in Nonlinear Science and Numerical Simulation, Vol. 89, p. 105288, 2020.

[7] A. Gusso, S. Ujevic and R. L. Viana. Strong chaotification and robust chaos in the Duffing oscillator induced by two-frequency excitation, Nonlinear Dynamics, Vol. 103, No. 2, 1955-1967, 2021.

[8] G. R. Liu and T. Y. Wu. Numerical solution of Duffing equation by the Laplace decomposition algorithm, Journal of Sound and Vibration, Vol. 237, No. 5, 805-817, 2000.

[9] E. Yusufo˘glu. Numerical solution for differential equations of Duffing type non-linearity using the generalized differential quadrature rule, Applied Mathematics and Computation, Vol. 177, No. 2, 572-580, 2006.

[10] J. A. Rad, S. Kazem and K. Parand. A numerical solution of the nonlinear controlled Duffing oscillator by radial basis functions, Computers & Mathematics with Applications, Vol. 64, No. 6, 2049-2065, 2012.

[11] M. Kami´nski and A. Corigliano. Numerical solution of the Duffing equation with random coefficients, Meccanica, Vol. 50, No. 7, 1841-1853, 2015.

[12] A. Azimi and M. Azimi. Computing Simulation of the Generalized Duffing Oscillator Basedon EBM and MHPM, Mechanics and Mechanical Engineering, Vol. 20, No. 4, 595–604, 2016.

[13] R. Najafi and B. Nemati Saray. Numerical solution of the forced Duffing equations using Legendre multiwavelets, Computational Methods for Differential Equation, Vol. 5, No. 1, 43–55, 2017.

[14] A. F. N. Rasedee, N. Ishak, S. R. Hamzah, H. M. Ijam, M. Suleiman, Z. B. Ibrahim, M. H. Abdul Sathar, N. A Ramli and N. S. Kamaruddin. Variable order variable stepsize algorithm for solving nonlinear Duffng oscillator, Journal of Physics: Conference Series, Vol. 890, p. 012045, 2017.

[15] G. Zhang and Z. Wu. Homotopy analysis method for approximations of Duffing oscillator with dual frequency excitations, Chaos, Solitons & Fractals, Vol. 127, 342-353, 2019.

[16] J. H. He and Y. O. El-Dib. The reducing rank method to solve third order Duffing equation with the homotopy perturbation, Numerical Methods for Partial Differential Equations, Vol. 37 No. 2, 1800– 1808, 2021.

[17] C. W. Gear. The numerical integration of ordinary differential equations, Mathematics of Computation, Vol. 21, No. 98, 146-156, 1967.

[18] F. T. Krogh. On testing a subroutine for the numerical integration of ordinary differential equations. Journal of the ACM, Vol. 20, No. 4, 545-562, 1973.

[19] J. D. Lambert. Computational methods in ordinary differential equations, Wiley, New York, 1973.

[20] G. Hall and J. M. Watt. Modern numerical methods for ordinary differential equations, Oxford University Press, London, 1976.

[21] M. B. Suleiman. Generalized multistep Adams and backward differentiation methods for the solution of stiff and non-stiff ordinary differential equations, Ph.D. thesis, University of Manchester, 1979.

[22] M. Suleiman. Solving nonstiff higher order ODEs directly by the direct integration method, Applied Mathematics and Computation, Vol. 33, No. 3, 197-219, 1989.

[23] A. F. N. Rasedee, N. Ishak, S. R. Hamzah, H. Mohd Ijam, M. Suleiman, Z. B. Ibrahim, M. H. A. Sathar, N. A. Ramli and N. S. Kamaruddin. Variable order variable stepsize algorithm for solving nonlinear Duffing oscillator, Journal of Physics: Cnf. Series, Vol. 890, 012045, 2017.

[24] A. F. N. Rasedee, H. Mohd Ijam, M. H. A. Sathar, N. Ishak, S. R. Hamzah, M. Sahrim and I. Ismail. Solution for nonlinear Riccati equation by block method, AIP Conference Proceedings, Vol. 1974, p. 020071, 2018.

[25] H. Mohd Ijam, Z. B. Ibrahim, N. Senu, M. Suleiman and A. F. N. Rasedee. Order and stability of 2-point block backward difference method, AIP Conference Proceedings, Vol. 1974, p. 020054, 2018.

[26] Z. B. Ibrahim, N. Zainuddin, K. I. Othman, M. Suleiman and I. S M. .Zawawi. Variable Order Block Method for Solving Second Order Ordinary Differential Equations, Sains Malaysiana, Vol. 48. No.8, 1761-1769, 2019.

[27] O. Adeyeye and Z. Omar. Implicit five-step block method with generalised equidistant points for solving fourth order linear and non-linear initial value problems, Ain Shams Engineering Journal, Vol. 10, No. 2, 881-889, 2019.

[28] A. I. Asnor, S. A. Mohd Yatim and Z. B. Ibrahim. Solving Directly Higher Order Ordinary Differential Equations by Using Variable Order Block Backward Differentiation Formulae, Symmetry, Vol. 11, No. 10, 1289, 2019.

[29] N. Mohd Nasir, Z. Abdul Majid, F. Ismail and N. Bachok. Direct integration of the third-order two point and multipoint Robin type boundary value problems, Mathematics and Computers in Simulation, Vol. 182, No. 1, 411–427, 2021.

[30] A. F. N. Rasedee, M. H. A. Sathar, S. R. Hamzah, N. Ishak, K. I. Othman, K. I. Approximating non linear higher order ODEs by a three point block algorithm, Plos one, Vol. 16, No. 2, pp.e0246904, 2021.

[31] A. F. N. Rasedee, M. H. A. Sathar, S. R. Hamzah, N. Ishak, T. J. Wong, L. F. Koo and S. N. I. Ibrahim. Two-point block variable order step size multistep method for solving higher order ordinary differential equations directly, Journal of King Saud University-Science, Vol. 33, No. 2, pp.101376, 2021.

[32] M. Suleiman, Z. B. Ibrahim and Rasedee, A. F. N. Solution of higher-order ODEs using backward difference method, Mathematical Problems in Engineering, Vol. 2011, 2011.

[33] H. Mohd Ijam, M. Suleiman, A. F. N. Rasedee, N. Senu, A. Ahmadian and S. Salahshour. Solving nonstiff higher-order ordinary differential equations using 2-point block method directly, Abstract and Applied Analysis, Vol. 2014, 2014.

[34] H. A. Watts and L. F. Shampine. A-stable block implicit one-step methods, BIT Numerical Mathematics, Vol. 12, No. 2, 252–266, 1972.

[35] S. Ola Fatunla. Block methods for second order ODEs, International Journal of Computer Mathematics, Vol. 41, No. 1, 55–63, 1991.

[36] K. Tabatabaei and E. Gunerhan. Numerical solution of Duffing equation by the differential transform method, Appl. Math. Inf. Sci. Letter, Vol. 2, No. 1, 1–6, 2014.

[37] M. Khalid, E. Sultana, U. Arshad and M. Shoaib. A comparison between new iterative solutions of nonlinear oscillator equation, International Journal of Computer Applications, Vol. 128, No. 4, 1–5, 2015.

[38] M. B. Suleiman. Some necessary conditions for convergence of the GBDF methods, Mathematics of Computation, Vol. 60, No. 202, 635–649, 1993.

[39] A. F. N. Rasedee, M. H. A. Sathar, N. Ishak, N. S. Kamarudin, M. A. Nazri, N. A. Ramli, I. Ismail and M. Sahrim. Solution for nonlinear Duffing oscillator using variable order variable stepsize block method, Matematika, Vol. 33, No. 2, 165–176, 2017.

[40] A. F. N. Rasedee. Direct method using backward difference for solving higher order ordinary differential equations, Universiti Putra Malaysia, Selangor, 2009.