CHAPTER 5

NUMERICAL RESULTS

5.1 Introduction

In this chapter, we considered the calculation of the special logarithmic singular integral equations of order *k* (LogSIEk) and special singular integral equations (SSIEk) in one dimension. It was shown that the unknown function can be written in terms of Extended Chebyshev polynomials. In doing so, we obtained a system of linear equations and the singular integral equations are formulated as a system of linear equations with unknown coefficients. We have tested the proposed methods of Chapters 3 and 4 and on some LogSIEk and SSIEk.

Several methods have been recently developed for the numerical solution of the CSIEs. Specifically, Dezhbord et al. (2016) investigate the numerical solution of various cases (Case (I), Case (II), Case (III) and Case (IV)) of CSIEs using reproducing kernel Hilbert space method. Eshkuvatov et al. (2009) considered a method based on collocation method; they studied efficient approximate methods for solving CSIEs of the first kind with four cases (Case (I), Case (II), Case (III) and Case (IV)). We applied the same example and compared the numerical results obtained with those by various methods to make comparison between the proposed method and the methods presented by Dezhbord et al. (2016) and Eshkuvatov et al. (2009).

All the numerical calculations reported in this chapter have been performed on the MATLAB programming using programmes written in MATLAB code, some of these code are found in Appendix, a great deal of effort has been made in the development of MATLAB code for the approximate solution of LogSIEk and SSIEk.

5.2 Special Logarithmic Singular Integral Equations of Order k

This section reports on two numerical experiments homogeneous and inhomogeneous cases for the first and second kinds LogSIEk to study the accuracy and the performance of the proposed adaptive strategy method. All experiments have been conducted using MATLAB. All exact solutions are always known. Two examples were performed with $n = 5$ and $n = 10$. We restrict the presentation to the simplest case $k = 1, 3, 5$ respectively. The right-hand side for the Galerkin method is always computed as explained in Chapters 3 and 4.

Example 1 *Consider a linear LogSIEk of the first kind:*

$$\int_{-1}^{1} \phi(x) \log|x^5 - t^5| dx = -\frac{\pi}{5} t^5, \qquad t \in (-1,1) \tag{5.1}$$

The exact solution of Eq. (5.1) *is:*

$$\phi(x) = \frac{x^9}{\sqrt{1-x^{10}}}, \qquad x \in (-1,1)$$

The approximation solution of Eq. (5.1) *has a form:*

$$\phi_5(t) = a_0 Z^5_{(1,0)}(t) \frac{t^2}{\sqrt{1-t^6}} + \frac{t^2}{\sqrt{1-t^6}} \sum_{i=1}^{5} a_i Z^5_{(1,i)}(t) \tag{5.2}$$

We find the unknown coefficients:

$$a_i = [0, 1.0, 0, 0, 0, 0]$$

Here, the exact and approximation solutions coincide. The absolute errors between exact and approximation solutions equal zero.

Now, we reduce Eq. (5.1) **to the form**

$$\int_{-1}^{1} \phi(x) \log|x - t| dx = -\pi t, \qquad t \in (-1,1) \tag{5.3}$$

The exact solution of Eq. (5.3) *is:*

$$\phi(x) = \frac{x}{\sqrt{1-x^2}}, \qquad x \in (-1,1)$$

The approximation solution of Eq. (5.3) *has a form*

$$\phi_5(t) = a_0 T_0(t) \frac{1}{\sqrt{1-t^2}} + \frac{1}{\sqrt{1-t^2}} \sum_{i=1}^{5} a_i T_i(t) \tag{5.4}$$

where $T_i(t)$ *is Chebyshev polynomials of the first kind. We find the unknown coefficients:*

$$a_i = [0, 1.0, 0, 0, 0, 0]$$

The exact and approximation solutions coincide.

Example 2 *Consider inhomogeneous linear LogSIEk of the second kind (LogSIE3):*

$$\phi(t) = \frac{3t^5 + \pi t^3 \sqrt{1-t^6}}{3\sqrt{1-t^6}} + \int_{-1}^{1} \phi(x) \log|x^3 - t^3| dx, \quad t \in (-1, 1) \tag{5.5}$$

The exact solution of Eq. (5.5) *is:*

$$\phi(t) = \frac{t^5}{\sqrt{1-t^6}}, \quad t \in (-1, 1)$$

The approximation solution of Eq. (5.5) *has a form:*

$$\phi_5(t) = a_0 Z^3_{(1,0)}(t) \frac{t^2}{\sqrt{1-t^6}} + \frac{t^2}{\sqrt{1-t^6}} \sum_{i=1}^{5} a_i Z^3_{(1,i)}(t) \tag{5.6}$$

We find the the unknown coefficients by using MATLAB code:

$$a_i = [0, 1, 0, 3.729E-17, 0, 0]$$

Table 5.1: Example 2: Absolute Error for different values of n

x	$n = 5$	$n = 10$
0.9	0.000000000000000 E+00	0.000000000000000 E+00
0.8	5.551115123125783 E-17	0.000000000000000 E+00
0.7	2.775557561562891 E-17	0.000000000000000 E+00
0.6	1.387778780781446 E-17	0.000000000000000 E+00
0.5	0.000000000000000 E+00	0.000000000000000 E+00
0.4	1.734723475976807 E-18	0.000000000000000 E+00
0.3	4.336808689942018 E-19	0.000000000000000 E+00
0.2	5.421010862427522 E-20	0.000000000000000 E+00
0.1	1.694065894508601 E-21	0.000000000000000 E+00
0	0.000000000000000 E+00	0.000000000000000 E+00
-0.1	1.694065894508601 E-21	0.000000000000000 E+00
-0.2	5.421010862427522 E-20	0.000000000000000 E+00
-0.3	4.336808689942018 E-19	0.000000000000000 E+00
-0.4	1.734723475976807 E-18	0.000000000000000 E+00
-0.5	0.000000000000000 E+00	0.000000000000000 E+00
-0.6	1.387778780781446 E-17	0.000000000000000 E+00
-0.7	2.775557561562891 E-17	0.000000000000000 E+00
-0.8	5.551115123125783 E-17	0.000000000000000 E+00
-0.9	0.000000000000000 E+00	0.000000000000000 E+00

Eq. (5.5) **can be reduced to the form**:

$$\phi(t) = \frac{t}{\sqrt{1-t^2}} + \pi t + \int_{-1}^{1} \phi(x) \log|x-t|\,dx, \quad t \in (-1,1) \tag{5.7}$$

The exact solution of Eq. (5.7) *is:*

$$\phi(t) = \frac{t}{\sqrt{1-t^2}}, \qquad t \in (-1,1)$$

The approximation solution of Eq. (5.7) *has a form*

$$\phi_5(t) = a_0 T_0(t) \frac{1}{\sqrt{1-t^2}} + \frac{1}{\sqrt{1-t^2}} \sum_{i=1}^{5} a_i T_i(t) \tag{5.8}$$

We find the unknown coefficients:

$$a_i = [0, 1, 0, -5.551E-17, 0, -5.551E-17]$$

Table 5.2: Absolute Error for different values of n of Eq. (5.7) (after reduction order)

x	$n = 5$	$n = 10$
0.9	0.000000000000000 E+00	0.000000000000000 E+00
0.8	0.000000000000000 E+00	-2.220446049250313 E-16
0.7	1.110223024625157 E-16	2.220446049250313 E-16
0.6	1.110223024625157 E-16	1.110223024625157 E-16
0.5	1.110223024625157 E-16	1.110223024625157 E-16
0.4	0.000000000000000 E+00	0.000000000000000 E+00
0.3	0.000000000000000 E+00	0.000000000000000 E+00
0.2	2.775557561562891 E-17	0.000000000000000 E+00
0.1	1.387778780781446 E-17	1.387778780781446 E-17
0	6.798155367234457 E-33	6.798155367234457 E-33
-0.1	1.387778780781446 E-17	1.387778780781446 E-17
-0.2	2.775557561562891 E-17	0.000000000000000 E+00
-0.3	0.000000000000000 E+00	0.000000000000000 E+00
-0.4	0.000000000000000 E+00	0.000000000000000 E+00
-0.5	1.110223024625157 E-16	1.110223024625157 E-16
-0.6	1.110223024625157 E-16	1.110223024625157 E-16
-0.7	1.110223024625157 E-16	2.220446049250313e E-16
-0.8	0.000000000000000 E+00	2.220446049250313 E-16
-0.9	0.000000000000000 E+00	0.000000000000000 E+00

5.3 Special Singular Integral Equations

This section considers three numerical experiments to study the accuracy and the performance of the proposed adaptive strategy method. All experiments have been conducted using MATLAB. The exact solution of all examples are always known. Three examples were performed with $n = 5$. We restrict the presentation to the simplest case $k = 1, 3, 5$ respectively. We solve all examples in four cases (**Case (I)** - **Case (IV)**) and make comparison test between of them and the exact solutions. The computational results are presented.

Example 3 has been solved by several researchers using approaches, see Eshkuvatov et al. (2009) and Dezhbord et al. (2016). Therefore, we will compare results with those.

Example 3 *Consider the singular integral equation (SIE1), (Eshkuvatov et al. (2009), Dezhbord et al. (2016)):*

$$\int_{-1}^{1} \frac{\phi(t)}{t-x} dt = x^4 + 5x^3 + 2x^2 + x - \frac{11}{8}, \quad x \in (-1, 1) \tag{5.9}$$

Case (I): The solution is unbounded at both end-points $x = \pm 1$:

The exact solution of Eq. (5.9) *is:*

$$\phi(t) = \frac{1}{\pi\sqrt{1-t^2}}(t^5 + 5t^4 + \frac{3}{2}(t^3 - t^2) - \frac{5}{2}t - \frac{9}{8})$$

The approximation solution of Eq. (5.9) *has a form:*

$$\phi_5(t) = a_0 T_0(t) \frac{1}{\sqrt{1-t^2}} + \frac{1}{\sqrt{1-t^2}} \sum_{i=1}^{5} a_i T_i(t) \tag{5.10}$$

where $T_i(t)$ is Chebyshev polynomials of the first kind. Applying the proposed method, we find the unknown coefficients:

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.23873241463784301880823018263044 \\ 0.55704230082163371055253709280436 \\ 0.21883804675135610057421100074457 \\ 0.19894367886486918234019181885877 \\ 0.019894367886486918234019181885877 \end{bmatrix}$$

Table 5.3: Example 3 Case (I): Absolute Error compared with Eshkuvatov et al. (2009)

x	$n = 5$	$n = 20$
-0.95	2.220446049250313 E-17	8.881784197001252 E-16
-0.9	6.661338147750939E-17	-6.661338147750939 E-16
-0.7	2.775557561562891E-17	7.494005416219807 E-16
-0.5	6.938893903907228E-17	-5.551115123125783 E-16
-0.3	5.551115123125783E-17	2.220446049250313 E-16
-0.1	5.551115123125783E-17	-3.330669073875470 E-16
0.0	0.000000000000000E+00	1.665334536937735 E-16
0.1	5.551115123125783E-17	5.551115123125783 E-16
0.3	1.110223024625157E-16	0.000000000000000E+00
0.5	1.110223024625157E-16	-7.771561172376096E-16
0.7	0.000000000000000E+00	0.000000000000000E+00
0.9	3.885780586188048E-16	9.436895709313831E-16
0.95	4.440892098500626E-16	8.881784197001252 E-16

The absolute errors between exact and approximation solutions compared with results are given by Table 5.3.

Case (II): The solution is bounded at both end-points $x = \pm 1$:

The exact solution of Eq. (5.9) *is*

$$\phi(t) = -\frac{1}{\pi}\sqrt{1-t^2}(t^3 + 5t^2 + \frac{5}{2}t + \frac{7}{2})$$

The approximation solution of Eq. (5.9) *has a form:*

$$\phi_5(t) = a_0\sqrt{1-t^2}U_0(t) + \sqrt{1-t^2}\sum_{i=1}^{5} a_i U_i(t) \qquad (5.11)$$

where $U_i(t)$ is Chebyshev polynomials of the second kind. Applying proposed method, the unknown coefficients are found:

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} 0 \\ -1.5119719593730057857854578233261 \\ -0.47746482927568603761646036526088 \\ -0.3978873577297383646803836377174 \\ -0.03978873577297383646803836377174 \\ 0 \end{bmatrix}$$

Table 5.4: Example 3 Case (II): Absolute Error compared with Eshkuvatov et al. (2009)

x	$n=5$	$n=20,$
-0.95	5.551115123125783E-17	1.665334536937735 E-16
-0.9	0.000000000000000E+00	3.330669073875470 E-16
-0.7	4.440892098500626E-16	17.76356839400250 E-16
-0.5	5.551115123125783E-16	5.551115123125783 E-16
-0.3	4.440892098500626E-16	2.220446049250313 E-16
-0.1	2.220446049250313E-16	2.220446049250313 E-16
0.0	0.000000000000000E+00	6.661338147750939 E-16
0.1	2.220446049250313E-16	0.000000000000000E+00
0.3	2.220446049250313E-16	6.661338147750939E-16
0.5	2.220446049250313E-16	8.881784197001252E-16
0.7	2.220446049250313E-16	0.000000000000000E+00
0.9	2.220446049250313E-16	2.220446049250313E-16
0.95	2.220446049250313E-16	4.440892098500626E-16

Table 5.5: Example 3 Case (II): Absolute Error compared with Dezhbord et al. (2016)

x	$n=5$	$n=200,$
-0.9	0.000 E+0	3.298 E-7
-0.6	4.441 E-16	2.689 E-8
-0.3	4.441 E-16	6.313 E-8
0.0	0.000 E+0	8.814E-8
0.3	2.220 E-16	1.193E-7
0.6	2.220 E-16	2.078E-7
0.9	2.220 E-16	4.104E-7

Case (III): The solution is bounded at the point $x=-1$:

The exact solution of Eq. (5.9) *is*

$$\phi(t)=\frac{1}{\pi}\sqrt{\frac{1+t}{1-t}}(t^4+4t^3-\frac{5}{2}t^2+t-\frac{7}{2}) \tag{5.12}$$

The approximation solution of Eq. (5.9) *has a form*

$$\phi_5(t)=a_0\sqrt{\frac{1+t}{1-t}}V_0(t)+\sqrt{\frac{1+t}{1-t}}\sum_{i=1}^{5}a_iV_i(t) \tag{5.13}$$

where $V_i(t)$ is Chebyshev polynomials of the third kind. Apply proposed method, the unknown coefficients are found:

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} -0.7559859796862928860150532273\,11 \\ 0.51725356504845010352511280871113 \\ 0.03978873577318375875933043062104\,8 \\ 0.17904931097817231022695239062159 \\ 0.01989436788669689606945389925840\,6 \\ -2.099890478883082430439340790E-13 \end{bmatrix}$$

Table 5.6: Example 3 Case (III): Absolute Error compared with Dezhbord et al. (2016)

x	$n=5$	$n=200$
-0.9	1.599 E-12	9.949 E-9
-0.6	1.130 E-13	4.933 E-8
-0.3	2.860 E-13	1.892 E-9
0.0	0.000 E+00	9.718 E-10
0.3	2.860 E-13	9.611 E-10
0.6	1.137 E-14	7.964 E-7
0.9	1.600 E-12	1.154 E-7

The absolute errors between exact and approximation solutions with compared results are given by Table 5.6.

Case (IV): The solution is bounded at the point $x = 1$:
The exact solution of Eq. (5.9) is

$$\phi(t) = -\frac{1}{\pi}\sqrt{\frac{1-t}{1+t}}(t^4 + 6t^3 - \frac{15}{2}t^2 + 6t + \frac{7}{2}) \tag{5.14}$$

The approximation solution of Eq. (5.9) has a form

$$\phi_5(t) = -a_0\sqrt{\frac{1-t}{1+t}}W_0(t) - \sqrt{\frac{1-t}{1+t}}\sum_{i=1}^{5} a_i W_i(t) \tag{5.15}$$

where $W_i(t)$ is Chebyshev polynomials of the fourth kind. The unknown coefficients are:

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} 0.75598597968613712172469831784838 \\ 0.99471839432398001523694111989639 \\ 0.43767609350234643050825411592086 \\ 0.21883804675099033376639567904931 \\ 0.019894367886121169103352812612684 \\ -3.657695932040300321376818072E-13 \end{bmatrix}$$

Table 5.7: Example 3 Case (IV): Absolute Error compared with Dezhbord et al. (2016)

x	$n=5$	$n=200,$
-0.9	9.184 E-13	2.594 E-8
-0.6	6.506 E-14	3.486 E-7
-0.3	1.639 E-13	8.386 E-9
0.0	0.000 E+00	7.021 E-9
0.3	1.636 E-13	7.156 E-9
0.6	6.528 E-14	2.385 E-8
0.9	9.199 E-13	1.901 E-8

The absolute errors between exact and approximation solutions compared with results in Dezhbord et al. (2016) are given by Table 5.7.

Example 4 *Consider the singular integral equation $SSIE_3$:*

$$\int_{-1}^{1}\frac{f(x)}{x^3-t^3}dx = 8t^{12}-8t^6+1, \qquad t\in(-1,1) \tag{5.16}$$

Case (I): The solution is unbounded at both end-points $x=\pm1$:
The exact solution of Eq. (5.16) *is:*

$$f(t)=\frac{3t^2}{\pi\sqrt{1-t^6}}(8t^{15}-12t^9+4t^3) \tag{5.17}$$

The approximation solution of Eq. (5.16) *has a form*

$$f_5(t)=a_0Z^3_{(1,0)}(t)\frac{t^2}{\sqrt{1-t^6}}+\frac{t^2}{\sqrt{1-t^6}}\sum_{i=1}^{5}a_iZ^3_{(1,i)}(t) \tag{5.18}$$

The unknown coefficients are found:

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.47746482927568602599735201099929 \\ 0 \\ 0.47746482927568602599735201099929 \end{bmatrix}$$

Table 5.8: Example 4 Case (I): Absolute Error

x	$n=5$	$n=7$	$n=9$
0.9	5.828670879282072 E-16	5.828670879282072 E-16	5.828670879282072 E-16
0.7	1.110223024625157E-16	1.110223024625157E-16	1.110223024625157E-16
0.5	1.249000902703301 E-16	1.249000902703301 E-16	1.249000902703301 E-16
0.3	2.081668171172169 E-17	2.081668171172169 E-17	2.081668171172169 E-17
0.1	4.208059681959364e E-18	4.208059681959364e E-18	4.208059681959364e E-18
0.0	0.0000000000000000 E+00	0.0000000000000000 E+00	0.0000000000000000 E+00
-0.1	4.065758146820642 E-19	4.065758146820642 E-19	4.065758146820642 E-19
-0.3	2.081668171172169 E-17	2.081668171172169 E-17	2.081668171172169 E-17
-0.5	1.110223024625157-16	1.110223024625157-16	1.110223024625157-16
-0.7	5.551115123125783 E-17	5.551115123125783 E-17	5.551115123125783 E-17
-0.9	4.163336342344337 E-16	4.163336342344337 E-16	4.163336342344337 E-16

The absolute errors between exact and approximation solutions are given by Table 5.8.

Eq. (5.16) can be reduced to the form:

$$\int_{-1}^{1} \frac{f(x)}{x-t}dx = 8t^4 - 8t^2 + 1, \qquad t \in (-1,1) \tag{5.19}$$

Case (I): The solution is unbounded at both end-points $x = \pm 1$ (after reduction order):

The exact solution of Eq. (5.19) is:

$$f(t) = \frac{1}{\pi\sqrt{1-t^2}}(8t^5 - 12t^3 + 4t) \tag{5.20}$$

The approximation solution of Eq. (5.19) has a form

$$f_5(t) = a_0 T_0(t)\frac{1}{\sqrt{1-t^2}} + \frac{1}{\sqrt{1-t^2}}\sum_{i=1}^{5} a_i T_i(t) \tag{5.21}$$

where $T_i(t)$ is Chebyshev polynomials of the first kind. The unknown coefficients are found:

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.159154943091895345608222100096376 \\ 0 \\ 0.159154943091895345608222100096376 \end{bmatrix}$$

Table 5.9: Example 4 Case (I): Absolute Error of Eq. (5.19) (after reduction order)

x	$n = 5$	$n = 7$	$n = 9$
0.9	7.216449660063518 E-16	7.216449660063518 E-16	7.216449660063518 E-16
0.7	2.046973701652632E-16	2.046973701652632E-16	2.046973701652632E-16
0.5	1.110223024625157 E-16	1.110223024625157 E-16	1.110223024625157 E-16
0.3	1.110223024625157 E-16	1.110223024625157 E-16	1.110223024625157 E-16
0.1	2.775557561562891 E-17	2.775557561562891 E-17	2.775557561562891 E-17
0.0	7.796343665038752 E-17	7.796343665038752 E-17	7.796343665038752 E-17
-0.1	2.220446049250313E-16	2.220446049250313E-16	2.220446049250313E-16
-0.3	0.000000000000000 E+00	0.000000000000000 E+00	0.000000000000000 E+00
-0.5	3.885780586188048E-16	3.885780586188048E-16	3.885780586188048E-16
-0.7	3.712308238590367 E-16	3.712308238590367 E-16	3.712308238590367 E-16
-0.9	3.885780586188048 E-16	3.885780586188048 E-16	3.885780586188048 E-16

The absolute errors between exact and approximation solutions are given by Table 5.9.

Case (II) : The solution is bounded at both end-points $x = \pm 1$:

The exact solution of Eq. (5.16) *is:*

$$f(t) = \frac{3}{\pi} t^2 \sqrt{1 - t^6} (8t^9 - 4t^3) \tag{5.22}$$

The approximation solution of Eq. (5.9) *has a form:*

$$\phi_5(t) = t^2 \sqrt{1 - t^6} a_0 Z^3_{(2,0)}(t) + t^2 \sqrt{1 - t^6} \sum_{i=1}^{5} a_i Z^3_{(2,i)}(t) \tag{5.23}$$

The unknown coefficients are found:

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.95492965855137199610918857921504 \\ 0 \end{bmatrix}$$

Table 5.10: Example 4 Case (II): Absolute Error

x	$n = 5$	$n = 7$	$n = 9$
0.9	3.053113317719181 E-16	3.191891195797325 E-16	3.191891195797325 E-16
0.7	5.551115123125783 E-17	5.551115123125783 E-17	5.551115123125783 E-17
0.5	8.326672684688674 E-17	8.326672684688674 E-17	8.326672684688674 E-17
0.3	1.908195823574488 E-17	1.908195823574488 E-17	1.908195823574488 E-17
0.1	4.214835945537399 E-18	4.214835945537399 E-18	4.214835945537399 E-18
0.0	0.000000000000000 E+00	0.000000000000000 E+00	0.000000000000000 E+00
-0.1	4.133520782600986 E-19	4.336808689942018 E-19	4.336808689942018 E-19
-0.3	2.428612866367530 E-17	2.428612866367530 E-17	2.428612866367530 E-17
-0.5	1.387778780781446E-17	1.387778780781446E-17	1.387778780781446E-17
-0.7	5.551115123125783 E-17	5.551115123125783 E-17	5.551115123125783 E-17
-0.9	7.771561172376096 E-16	7.771561172376096 E-16	7.771561172376096 E-16

The absolute errors between exact and approximation solutions are given by Table 5.10.

Case (II) : The solution is bounded at both end-points $x = \pm 1$ (after reduction order):

The exact solution of Eq. (5.19) *is:*

$$f(t) = \frac{1}{\pi}\sqrt{1-t^2}(8t^3 - 4t) \tag{5.24}$$

The approximation solution of Eq. (5.19) *has a form:*

$$\phi_5(t) = \sqrt{1-t^2}a_0U_0(t) + \sqrt{1-t^2}\sum_{i=1}^{5} a_iU_i(t) \tag{5.25}$$

where $U_i(t)$ is Chebyshev polynomials of the second kind. The unknown coefficients are found:

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.31830988618379069121644420192752 \\ 0 \end{bmatrix}$$

Table 5.11: Example 4 Case (II): Absolute Error of Eq. (5.19) (after reduction order)

x	$n = 5$	$n = 7$	$n = 9$
0.9	1.110223024625157 E-16	1.110223024625157 E-16	1.110223024625157 E-16
0.7	7.285838599102590 E-17	7.285838599102590 E-17	7.285838599102590 E-17
0.5	5.551115123125783 E-17	5.551115123125783 E-17	5.551115123125783 E-17
0.3	1.110223024625157 E-16	1.110223024625157 E-16	1.110223024625157 E-16
0.1	5.551115123125783 E-17	5.551115123125783 E-17	5.551115123125783 E-17
0.0	0.000000000000000 E+00	0.000000000000000 E+00	0.000000000000000 E+00
-0.1	5.551115123125783 E-17	5.551115123125783 E-17	5.551115123125783 E-17
-0.3	1.110223024625157 E-16	1.110223024625157 E-16	1.110223024625157 E-16
-0.5	2.220446049250313 E-16	2.220446049250313E-16	2.220446049250313E-16
-0.7	3.417405247674310 E-16	3.417405247674310 E-16	3.417405247674310 E-16
-0.9	1.665334536937735 E-16	1.665334536937735 E-16	1.665334536937735 E-16

Case (III): The solution is bounded at points $x = -1$:

The exact solution of Eq. (5.16) *is:*

$$f(t) = \frac{3}{\pi} t^2 \sqrt{\frac{1+t^3}{1-t^3}} (8t^{12} - 8t^9 - 4t^6 + 4t^3) \qquad (5.26)$$

The approximation solution of Eq. (5.16) *has a form:*

$$f_5(t) = a_0 \frac{t^2}{\pi} \sqrt{\frac{1+t^3}{1-t^3}} Z^3_{(3,0)}(t) + \frac{t^2}{\pi} \sqrt{\frac{1+t^3}{1-t^3}} \sum_{i=1}^{5} a_i Z^3_{(3,i)}(t) \qquad (5.27)$$

The unknown coefficients are found:

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} -0.0000000000000220389632900042202 \\ 0.0000000000000220285624413461330 6 \\ -0.0000000000000220216417245584317 \\ 0.477464829275465463265248899915 6 \\ 0.954929658551371996109188579215 04 \\ 0.000000000000037217162973165470334 \end{bmatrix}$$

Table 5.12: Example 4 Case (III): Absolute Error

x	$n = 5$	$n = 7$	$n = 9$
0.9	4.898859096158503 E-13	4.898859096158503 E-13	4.898859096158503 E-13
0.7	6.111777750561487 E-14	6.111777750561487 E-14	6.111777750561487 E-14
0.5	1.016686734800487 E-13	1.016686734800487 E-13	1.016686734800487 E-13
0.3	5.106678968580525 E-14	5.106678968580525 E-14	5.106678968580525 E-14
0.1	5.924968360989519 E-15	5.924968360989519 E-15	5.924968360989519 E-15
0.0	0.000000000000000 E+00	0.000000000000000 E+00	0.000000000000000 E+00
-0.1	5.933221850027565 E-15	5.933221850027565 E-15	5.933221850027565 E-15
-0.3	5.473399511402022 E-14	5.473399511402022 E-14	5.473399511402022 E-14
-0.5	1.464522947358660 E-13	1.464522947358660 E-13	1.464522947358660 E-13
-0.7	9.481304630298837 E-14	9.481304630298837 E-14	9.481304630298837 E-14
-0.9	1.987160436200952 E-13	1.987160436200952 E-13	1.987160436200952 E-13

Table 5.12 refers to the absolute errors between exact and approximation solutions.

Case (III) : The solution is bounded at points $x = -1$ (after reduction order):

The exact solution of Eq. (5.19) *is:*

$$f(t) = \frac{1}{\pi} \sqrt{\frac{1+t}{1-t}} (8t^4 - 8t^3 - 4t^2 + 4t) \qquad (5.28)$$

The approximation solution of Eq. (5.19) *has a form*

$$f_5(t) = a_0 \frac{1}{\pi}\sqrt{\frac{1+t}{1-t}}V_0(t) + \frac{1}{\pi}\sqrt{\frac{1+t}{1-t}}\sum_{i=1}^{5} a_i V_i(t) \tag{5.29}$$

where $V_i(t)$ is Chebyshev polynomials of the third kind. The unknown coefficients are found:

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} -0.000000000000047989612378746780 \\ 0.000000000000047978011939178823 \\ -0.000000000000047980296874245238 \\ -0.159154943091847411729133909830 \\ 0.1591549430918474117291339098301 \\ 0.0000000000000479498896614383248 \end{bmatrix}$$

Table 5.13: Example 4 Case (III): Absolute Error of Eq. (5.19) (after reduction order)

x	$n = 5$	$n = 7$	$n = 9$
0.9	2.096101070492296 E-13	2.096101070492296 E-13	2.096101070492296 E-13
0.7	6.327924295668197 E-14	6.327924295668197 E-14	6.327924295668197 E-14
0.5	2.775557561562891 E-15	2.775557561562891 E-15	2.775557561562891 E-15
0.3	3.747002708109903 E-14	3.747002708109903 E-14	3.747002708109903 E-14
0.1	8.798517470154366E-14	8.798517470154366E-14	8.798517470154366E-14
0.0	9.585601317401377 E-14	9.585601317401377 E-14	9.585601317401377 E-14
-0.1	8.813783036742961 E-14	8.813783036742961 E-14	8.813783036742961 E-14
-0.3	3.774758283725532 E-14	3.774758283725532 E-14	3.774758283725532 E-14
-0.5	5.551115123125783 E-17	5.551115123125783 E-17	5.551115123125783 E-17
-0.7	6.283688847030788 E-14	6.283688847030788 E-14	6.283688847030788 E-14
-0.9	2.098321516541546E-13	2.098321516541546E-13	2.098321516541546E-13

Table 5.13 refers to the absolute errors between exact and approximation solutions.

Case (IV):The solution is bounded at the point $x = 1$:

The exact solution of Eq. (5.16) *is*

$$f(t) = -\frac{3}{\pi}t^2\sqrt{\frac{1-t^3}{1+t^3}}(8t^{12} + 8t^9 - 4t^6 - 4t^3) \tag{5.30}$$

The approximation solution of Eq. (5.9) *has a form:*

$$\phi_5(t) = -a_0 \frac{t^2}{\pi}\sqrt{\frac{1-t^3}{1+t^3}} Z^3_{(4,0)}(t) - \frac{t^2}{\pi}\sqrt{\frac{1-t^3}{1+t^3}} \sum_{i=0}^{5} a_i Z^3_{(4,i)}(t) \tag{5.31}$$

The unknown coefficients are found:

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} -0.000000000000220374605057874 61922 \\ -0.00000000000022037460505787461922 \\ -0.00000000000022037460505787461922 \\ 0.47746482927546546326524889 99156 \\ 0.47746482927546557428755136243126 \\ -0.00000000000037208304455677211799 0 \end{bmatrix}$$

Table 5.14: Example 4 Case (IV): Absolute Error

x	$n=5$	$n=7$	$n=9$
0.9	1.979805208662810 E-13	1.979805208662810 E-13	1.979805208662810 E-13
0.7	9.459100169806334 E-14	9.459100169806334 E-14	9.459100169806334 E-14
0.5	1.465216836749050 E-13	1.465216836749050 E-13	1.465216836749050 E-13
0.3	5.475307707225596 E-14	5.475307707225596 E-14	5.475307707225596 E-14
0.1	5.927129989070912 E-15	5.927129989070912 E-15	5.927129989070912 E-15
0.0	0.000000000000000 E+00	0.000000000000000 E+00	0.000000000000000 E+00
-0.1	5.918009138294877 E-15	5.918009138294877 E-15	5.918009138294877 E-15
-0.3	5.108066747361306 E-14	5.108066747361306 E-14	5.108066747361306 E-14
-0.5	1.016547956922409 E-13	1.016547956922409 E-13	1.016547956922409 E-13
-0.7	6.089573290068984 E-14	6.089573290068984 E-14	6.089573290068984 E-14
-0.9	4.887756865912252 E-13	4.887756865912252 E-13	4.887756865912252 E-13

The absolute errors between exact and approximation solutions are given by Table 5.14.

Case (IV) : The solution is bounded at the point $x=1$ (after reduction order):

The exact solution of Eq. (5.19) *is:*

$$f(t) = -\frac{1}{\pi}\sqrt{\frac{1-t}{1+t}}(8t^4+8t^3-4t^2-4t) \tag{5.32}$$

The approximation solution of Eq. (5.19) *has a form*

$$f_5(t) = a_0 \frac{1}{\pi}\sqrt{\frac{1-t}{1+t}} W_0(t) + \frac{1}{\pi}\sqrt{\frac{1-t}{1+t}} \sum_{i=1}^{5} a_i W_i(t) \tag{5.33}$$

where $W_i(t)$ *is Chebyshev polynomials of the fourth kind. The unknown coefficients are found:*

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} -0.000000000000004796535691419559 \\ -0.000000000000004794285909200319 \\ -0.000000000000004796535691419559 \\ 0.159154943091847411729133909830 \\ 0.159154943091847383973558294201 \\ -0.000000000000004796535691419559 \end{bmatrix}$$

Table 5.15: Example 4 Case (IV): Absolute Error of Eq. (5.19) (after reduction order)

x	$n = 5$	$n = 7$	$n = 9$
0.9	2.098321516541546 E-13	2.098321516541546 E-13	2.098321516541546 E-13
0.7	6.301556498833350 E-14	6.301556498833350 E-14	6.301556498833350 E-14
0.5	0.000000000000000 E+00	0.000000000000000 E+00	0.000000000000000 E+00
0.3	3.752553823233029 E-14	3.752553823233029 E-14	3.752553823233029 E-14
0.1	8.795741912592803E-14	8.795741912592803E-14	8.795741912592803E-14
0.0	9.598250272923368 E-14	9.598250272923368 E-14	9.598250272923368 E-14
-0.1	8.779088567223425 E-14	8.779088567223425 E-14	8.779088567223425 E-14
-0.3	3.763656053479281 E-14	3.763656053479281 E-14	3.763656053479281 E-14
-0.5	3.330669073875470 E-16	3.330669073875470 E-16	3.330669073875470 E-16
-0.7	6.350996117898688 E-14	6.350996117898688 E-14	6.350996117898688 E-14
-0.9	2.087219286295294E-13	2.087219286295294E-13	2.087219286295294E-13

Table 5.15 refers to the absolute errors between exact and approximation solutions of Eq. (5.19) *(after reduction order).*

Example 5 *Consider the singular integral equation SSIE5:*

$$\int_{-1}^{1} \frac{\phi(x)}{x^5 - t^5} dx = 8t^{20} - 8t^{10} + 1, \qquad t \in (-1, 1) \tag{5.34}$$

Case (I): The solution is unbounded at both end-points $x = \pm 1$:

The exact solution of Eq. (5.34) *is:*

$$\phi(t) = \frac{5}{\pi}\frac{t^4}{\sqrt{1-t^{10}}}(8t^{25} - 12t^{15} + 4t^5) \tag{5.35}$$

The approximation solution of Eq. (5.34) *has a form*

$$\phi_5(t) = a_0 Z^5_{(1,0)}(t)\frac{t^4}{\pi\sqrt{1-t^{10}}} + \frac{t^4}{\pi\sqrt{1-t^{10}}}\sum_{i=1}^{5} a_i Z^5_{(1,i)}(t) \tag{5.36}$$

Use MATLAB code, the unknown coefficients are found:

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.79577471545947670999558668499881 \\ 0 \\ 0.79577471545947670999558668499881 \end{bmatrix}$$

Table 5.16: Example 5 Case (I): Absolute Error

x	$n=5$	$n=7$	$n=9$
0.9	1.110223024625157 E-16	1.110223024625157 E-16	1.110223024625157 E-16
0.7	1.942890293094024 E-16	1.942890293094024 E-16	1.942890293094024 E-16
0.5	1.734723475976807 E-18	1.734723475976807 E-18	1.734723475976807 E-18
0.3	4.065758146820642 E-18	4.065758146820642 E-18	4.065758146820642 E-18
0.1	2.541926022375454 E-20	2.541926022375454 E-20	2.541926022375454 E-20
0.0	0.0000000000000000 E+00	0.0000000000000000 E+00	0.0000000000000000 E+00
-0.1	3.797586192230950 E-20	3.797586192230950 E-20	3.797586192230950 E-20
-0.3	1.029992063861229 E-18	1.029992063861229 E-18	1.029992063861229 E-18
-0.5	6.938893903907228 E-18	6.938893903907228 E-18	6.938893903907228 E-18
-0.7	5.551115123125783 E-17	5.551115123125783 E-17	5.551115123125783 E-17
-0.9	0.0000000000000000 E+00	0.0000000000000000 E+00	0.0000000000000000 E+00

Eq. (5.34) **can be reduced to the** *Eq.* (5.16)

Case (I): The solution is unbounded at both end-points $x = \pm 1$ (after reduction order):

Eq. (5.20) and Eq. (5.21) represent the exact and approximation solutions respectively. Table 5.9 refers to the absolute error.

Case (II): The solution is bounded at both end-points $x = \pm 1$:

The exact solution of Eq. (5.34) is:

$$\phi(x) = \frac{5}{\pi} x^4 \sqrt{1 - x^{10}} (8x^{15} - 4x^5) \tag{5.37}$$

The approximation solution of Eq. (5.34) has a form

$$\phi_5(t) = a_0 t^4 \sqrt{1 - t^{10}} Z^5_{(2,0)}(t) + t^4 \sqrt{1 - t^{10}} \sum_{i=1}^{5} a_i Z^5_{(2,i)}(t) \tag{5.38}$$

The unknown coefficients are found:

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1.5915494309189536174604382105526 \\ 0 \end{bmatrix}$$

Table 5.17: Example 5 Case (II): Absolute Error

x	$n=5$	$n=7$	$n=9$
0.9	0.000000000000000 E+00	0.000000000000000 E+00	0.000000000000000 E+00
0.7	5.551115123125783 E-17	5.551115123125783 E-17	5.551115123125783 E-17
0.5	1.734723475976807 E-19	1.734723475976807 E-19	1.734723475976807 E-19
0.3	3.984442983884229 E-18	3.984442983884229 E-18	3.984442983884229 E-18
0.1	2.541595150130433E-20	2.541595150130433E-20	2.541595150130433E-20
0.0	0.000000000000000 E+00	0.000000000000000 E+00	0.000000000000000 E+00
-0.1	3.797834346414716 E-20	3.797834346414716 E-20	3.797834346414716 E-20
-0.3	1.138412281109780 E-18	1.138412281109780 E-18	1.138412281109780 E-18
-0.5	3.989863994746656 E-17	3.989863994746656 E-17	3.989863994746656 E-17
-0.7	5.551115123125783 E-17	5.551115123125783 E-17	5.551115123125783 E-17
-0.9	0.000000000000000 E+00	0.000000000000000 E+00	0.000000000000000 E+00

The absolute errors between exact and approximation solutions are given by Table 5.17

Case (II): The solution is bounded at both end-points $x = \pm 1$ (after reduction order):

Eq. (5.24) *and Eq.* (5.25) *represent the exact and approximation solutions respectively. Table 5.11 refers to the absolute error.*

Case (III): The solution is bounded at point $x = -1$:

The exact solution of Eq. (5.34) *is:*

$$\phi(t) = \frac{5}{\pi} t^4 \sqrt{\frac{1+t^5}{1-t^5}} (8t^{20} - 8t^{15} - 4t^{10} + 4t^5) \qquad (5.39)$$

The approximation solution of Eq. (5.34) *has a form*

$$\phi_5(t) = a_0 t^4 \sqrt{\frac{1+t^5}{1-t^5}} Z^5_{(3,0)}(t) + t^4 \sqrt{\frac{1+t^5}{1-t^5}} \sum_{i=1}^{5} a_i Z^5_{(3,i)}(t) \qquad (5.40)$$

The unknown coefficients are found:

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} -0.00000000000049554309547592779264 \\ 0.00000000000049574302729423914254 0 \\ -0.00000000000081234714538149777328 \\ -0.79577471545898048610467867547413 \\ 0.79577471545898015303777128792717 \\ 0.00000000000081161772380260358544 \end{bmatrix}$$

Table 5.18: Example 5 Case (III): Absolute Error

x	$n = 5$	$n = 7$	$n = 9$
0.9	4.408695630786497 E-13	4.408695630786497 E-13	4.408695630786497 E-13
0.7	5.803690861228006 E-14	5.803690861228006 E-14	5.803690861228006 E-14
0.5	5.619463228079269 E-14	5.619463228079269 E-14	5.619463228079269 E-14
0.3	7.981652448349474 E-15	7.981652448349474 E-15	7.981652448349474 E-15
0.1	9.909698929102953 E-17	9.909698929102953 E-17	9.909698929102953 E-17
0.0	0.0000000000000000 E+00	0.0000000000000000 E+00	0.0000000000000000 E+00
-0.1	9.915430380849271 E-17	9.915430380849271 E-17	9.915430380849271 E-17
-0.3	8.076953819310950 E-15	8.076953819310950 E-15	8.076953819310950 E-15
-0.5	6.607388247648060 E-14	6.607388247648060 E-14	6.607388247648060 E-14
-0.7	2.473021787352536 E-13	2.473021787352536 E-13	2.473021787352536 E-13
-0.9	3.763656053479281 E-14	3.763656053479281 E-14	3.763656053479281 E-14

The absolute errors between exact and approximation solutions are given by Table 5.18.

Case (III): The solution is bounded at point $x = -1$(after reduction order):

Eq. (5.28) *and Eq.* (5.29) *represent the exact and approximation solutions respectively. Table 5.13 shows the absolute error.*

Case (IV): The solution is bounded at the point $x = 1$:

The exact solution of Eq. (5.34) *is*

$$\phi(x) = -\frac{5}{\pi}\frac{t^4\sqrt{1-t^5}}{\sqrt{1+t^5}}(8t^{20} + 8t^{15} - 4t^{10} - 4t^5) \tag{5.41}$$

The approximation solution of Eq. (5.34) *has a form*

$$\phi_5(t) = -a_0 t^4 \sqrt{\frac{1-t^5}{1+t^5}} Z^5_{(4,0)}(t) - t^4 \sqrt{\frac{1-t^5}{1+t^5}} \sum_{i=0}^{5} a_i Z^5_{(4,i)}(t) \qquad (5.42)$$

The unknown coefficients are found:

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} -0.000000000000495795756565002577 \\ -0.0000000000004959890972244685351 \\ -0.0000000000008119956169097415126 \\ 0.79577471545898037508237621295848 \\ 0.79577471545898059712698113798979 \\ -0.0000000000008121801693574135823 \end{bmatrix}$$

Table 5.19: Example 5 Case (IV): Absolute Error

x	$n = 5$	$n = 7$	$n = 9$
0.9	3.796962744218035 E-14	3.796962744218035 E-14	3.796962744218035 E-14
0.7	2.474687121889474 E-13	2.474687121889474 E-13	2.474687121889474 E-13
0.5	6.608949498776440 E-14	6.608949498776440 E-14	6.608949498776440 E-14
0.3	8.077224869854072 E-15	8.077224869854072 E-15	8.077224869854072 E-15
0.1	9.918344620865357 E-17	9.918344620865357 E-17	9.918344620865357 E-17
0.0	0.00000000000000 E+00	0.00000000000000 E+00	0.00000000000000 E+00
-0.1	9.912612590092610 E-17	9.912612590092610 E-17	9.912612590092610 E-17
-0.3	7.982709545467648 E-15	7.982709545467648 E-15	7.982709545467648 E-15
-0.5	5.620157117469660 E-14	5.620157117469660 E-14	5.620157117469660 E-14
-0.7	5.792588630981754 E-14	5.792588630981754 E-14	5.792588630981754 E-14
-0.9	4.385380947269368 E-13	4.385380947269368 E-13	4.385380947269368 E-13

The absolute errors between exact and approximation solutions are given by Table 5.19

Case (IV): The solution is bounded at point $x = 1$(after reduction order):

Eq. (5.32) *and Eq.* (5.33) *represent the exact and approximation solutions respectively. Table 5.15 shows the absolute error.*

5.4 Conclusion

In this chapter, the proposed methods are built upon in Chapter 3 and 4. It was shown that the unknown function can be written with bases of Extended Chebyshev polynomials. The special logarithmic singular integral equations of order k (LogSIEk) and special singular integral equations (SSIEk) are formulated in a system of linear equations with unknown coefficients. The proposed method has been tested for many values of x in the interval $(-1,1)$, with some values lying close to the end points. All results show that the values of the approximation solution is good. The absolute errors between exact and approximation solutions with compared results are given in tabulated form. The results presented shows a very good convergence of the values use Extended Chebyshev polynomials (ECPs).

Section 5.2 reports on two examples of the homogeneous and inhomogeneous cases for the first and second kinds LogSIEk to study the accuracy and the performance of the proposed method. All exact solutions are always known. Two examples were performed with $n = 5, 10$. We restrict the presentation to the simplest case $k = 3, 5$ respectively. Also, we reduced example (1) and (2) to the reduction orders forms (Eq. (5.3) and (5.7)), then we found the exact and approximations solutions, absolute errors were presented with $n = 5, 10$ (before and after reduction orders forms).

Section 5.3 considers three examples to study the accuracy and the performance of the proposed method. The exact solution of all examples are always known. Example (**??**) was performed with $n = 5$, we solved in four cases (**Case (I)** - **Case (IV)**) and make comparison test between them and the exact solutions. Also, we applied the same example and compared the numerical results obtained with those by various methods to make comparison between the proposed method and the methods presented in Dezhbord et al. (2016) and Eshkuvatov et al. (2009).

Examples (4) and (5) were performed with $n = 5, 7$ and 9. We solved four cases (**Case (I)** - **Case (IV)**) and made comparison tests between them and the exact solutions. Moreover, we reduced it to the reduction order form (Eq. (5.16)), then we found the exact and approximation solutions in four cases (**Case (I)** - **Case (IV)**); absolute errors were presented with $n = 5, 7$ and 9 (before and after reduction orders). All experiments have been conducted using MATLAB