

Analysis of Different Boundary Conditions on Homogeneous One-Dimensional Heat Equation

M A Q M Sukry, M A H Mohamed, F Jamaluddin, A D H Badrolhisam and N Subani¹

¹Kolej GENIUS Insan, Universiti Sains Islam Malaysia, Bandar Baru Nilai, 71800, Nilai, Negeri Sembilan, Malaysia

Email: norazlina.subani@usim.edu.my

Abstract. An equation that involves derivatives of an unknown function with respect to two or more independent variables is known as a partial differential equation. To obtain the exact solution of a partial differential equation, an analytical solution is necessary. The suitable boundaries and initial conditions are required to solve these partial differential equations. Not only the equation, but also the boundary conditions depend on the general solution. Specifically, when coupled with varying sets of boundary conditions, these partial differential equations would have distinct general solutions. The homogeneous one-dimensional heat equation will be analytically solved in the current analysis using the process of variable separation. Our primary objective is to determine the flow characteristics of the heat equation with boundary conditions of different forms. The heat equation will be solved based on Dirichlet, Neumann and mixed boundary conditions to verify our goal. The findings have been carried out by different boundary conditions values, but the initial value remains the same. The results show that changes in the profile of the temperature depend on the types of boundary conditions. The flow characteristics of the heat equation can influence the boundary conditions.

Keyword: Homogeneous heat equation, one-dimensional flow, difference types of boundary conditions, analytical solution, separation of variables

Introduction

An equation consisting derivatives of an unknown function with respect to two or more independent variables is a partial differential equation. It is possible to classify the partial differential equation into three parabolic [1-2], hyperbolic and elliptic [3-4] forms. A parabolic partial differential equation that describes a wide family of scientific problems, such as acoustic propagation of the ocean and diffusion of heat. Hyperbolic partial differential equation which describes an elastic string's wave transformation and vibrations, while elliptic partial differential equation which describes the equation of Laplace.

A given partial differential equation can be solved by the use of numerical solution and analytical solution [5]. However, to ensure that the numerical solution is valid [6-9], it is important to understand the general theory of partial differential equations. In order to obtain the exact solution of the partial differential equation, an analytical solution is therefore required.

The suitable boundary and initial conditions are required to solve these partial differential equations. Not only the equation, but also the boundary conditions are depending on the general solution. In other words, when combined with various sets of boundary conditions, these partial differential equations would have distinct general solutions. Subani *et al.* [5] only consider the heat equation with Neumann boundary condition. However, they found that for the short rod, the heat temperature quickly converges to zero when compared to the long rod.

The heat equation propagates energy, which is highly non-physical, at an infinite speed. However, for all classical physics and engineering applications, the validity of the heat equation as a model of temperature evolution is still extremely strong. Temperature fluctuations are one of the main consequences of heat transfer, where the heating process raises the temperature while the cooling process reduces the temperature [10]. This ensures that there is no step shift in this process and that no work is performed on or by the system [11]. Javed [12] studies on sources of dry or moist heat. Hot water bottles, radiant heat and electric pads provide dry applications. Moist heat is known to be more penetrating than dry heat, although this is more due to the slower loss of heat from water-soaked materials than dry ones.

Sabaeian *et al.* [13] claimed that the role of temperature distribution is important in the measurement, simulation, and prediction of thermal effects from an Islamic perspective. The temperature is specific to the heat capacity, or the amount of energy needed to adjust the temperature of the substance. Due to physical or chemical changes, the heat measurement changes [14].

The verse in the Qur'an (*Surah Yassin: 80*) tells us about the development of fire from green trees [15-16]. In other words, fire can be produced through the use of green plants. The friction of two surface objects generates fire [17] in that life. The heat velocity in this study would be calculated from one zone of high to low heat. The heat velocity rate is dependent on the degree of velocity of friction between the two objects.

The homogeneous one-dimensional heat equation will be analytically solved in the current study using the method of variable separation. Our primary objective is to determine the flow characteristics of the heat equation with boundary conditions of different types. Based on Dirichlet, Neumann and mixed boundary conditions to verify our goals, the heat equation will be solved. The findings have been compared with various values of boundary conditions, but the initial condition remains the same.

Mathematical Formulation

The mathematical models are used to define the one-dimensional homogeneous heat boundary value problems that are presented below with Dirichlet boundary conditions. The heat equation is used to calculate the change in the temperature function, u over time t . Figure 1 shows a physical model of the heat equation problem.

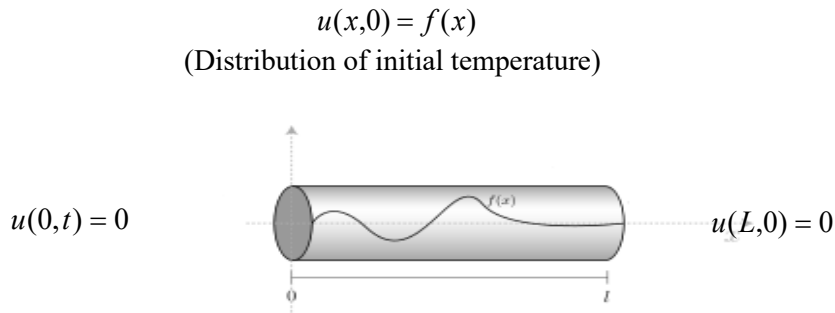


Figure 1. Physical model of heat equation problem with Dirichlet boundary condition.

Boundary Value Problem

To explain the one-dimensional homogeneous heat boundary value problems with Dirichlet boundary conditions, the mathematical models are used below. The heat equation is used to calculate the change in temperature function, u over time, t .

The general one-dimensional homogeneous heat equation can be described as:

$$u_t(x,t) = u_{xx}(x,t), \quad 0 < x < L, \quad t > 0 \tag{1}$$

where u is defined as heat temperature, x is space and t is time.

Boundary Conditions

Dirichlet, Neumann and mixed boundary conditions (left end insulated and right end insulated) are three types of boundary conditions that involve this problem.

The Dirichlet boundary conditions at the initial point, $x = 0$ and at the end point $x = 1$ are given by:

$$u(0,t) = 0, \quad t > 0 \quad \text{and} \quad u(1,t) = 0, \quad t > 0 \quad (2a)$$

The Neumann boundary conditions at the initial point, $x = 0$ and at the end point $x = 1$ are given by:

$$u_x(0,t) = 0, \quad t > 0 \quad \text{and} \quad u_x(1,t) = 0, \quad t > 0 \quad (2b)$$

The mixed boundary conditions at the initial point, $x = 0$ and at the end point $x = 1$ are given by:

(i) Left end insulated of the rod:

$$u_x(0,t) = 0, \quad t > 0 \quad \text{and} \quad u(1,t) = 0, \quad t > 0 \quad (2c)$$

(ii) Right end insulated of the rod:

$$u(0,t) = 0, \quad t > 0 \quad \text{and} \quad u_x(1,t) = 0, \quad t > 0 \quad (2d)$$

Initial Condition

The initial conditions at $t = 0$ is:

$$u(x,0) = x, \quad 0 < x < 1 \quad (3)$$

The heat conduction in a single-dimensional uniform rod of length one unit without internal heat sources, thermal diffusivity one, perfect lateral insulation and initial x when $0 < x < 1$ were defined by these mathematical models of Equations (1)-(3).

Analytical Solution

Using Crank-Nicolson [18], finite different method [19-21] or finite element method [22-23], most of the heat equation will be solved numerically. However, in order to obtain the exact solution of the partial differential equation, an analytical solution is required. Separation of Variables (SOV) is used to analytically address a partial differential equation (1). It is possible to divide a single partial differential equation into two ordinary differential equations, where only one independent variable remains for each equation.

Transformation of Partial Differential Equation into Separation of Variable Method

The partial differential equation (1) can be written in the form:

$$u(x,t) = X(x)T(t) \quad (4)$$

where

$$u_t(x,t) = X(x)T'(t) \quad \text{and} \quad u_{xx}(x,t) = X''(x)T(t) \quad (5)$$

Then, substituting equation (5) into equation (1) yields:

$$\begin{aligned} X(x)T'(t) &= X''(x)T(t) \\ \frac{X''(x)}{X(x)} &= \frac{T'(t)}{T(t)} = k \end{aligned} \quad (6)$$

Now, the two ordinary differential equations become the X -problem and T -problem where:

$$X\text{-problem: } X''(x) = kX(x) \quad (7a)$$

$$T\text{-problem: } T'(t) = kT(t) \quad (7b)$$

X-Problem by Sturm-Liouville Solver

To solve the heat equation (1), the Dirichlet boundary conditions (2a) will be used. Now, the boundary conditions can be determined as:

$$u(0,t) = X(0)T(t), \text{ where } X(0) = 0, \quad T(t) \neq 0 \quad (8a)$$

and

$$u(1,t) = X(1)T(t), \text{ where } X(1) = 0, \quad T(t) \neq 0 \quad (8b)$$

Then, the X -problem will be solved. To solve the boundary value problem (BVP), there are Three cases are required to consider when solving the boundary value problem (BVP), which are $\lambda = 0$, $\lambda > 0$ and $\lambda < 0$, $\lambda = -\lambda^2$; $\lambda \neq 0$. The trivial solution occurs when Eigen values, $\lambda = 0$ and $\lambda > 0$, while the non-trivial solution, $\lambda < 0$, $\lambda = -\lambda^2$; $\lambda \neq 0$. The trivial solution of the equation is called the constant zero solution. Thus, it is important to consider non-zero (non-trivial) solutions. Assume that as a constant $k = -\lambda$ of separation. The negative sign may be either positive or negative or even zero.

The equation (7a) can now be written as $X''(x) = -\lambda^2 X(x)$. Find the two-point boundary value problem's eigenvalues and Eigen functions. the characteristic equation becomes $m^2 + \lambda^2 = 0$, which has complex conjugate roots $m = \pm \lambda i$, is solved by this equation. The solution thus becomes:

$$X(x) = C_1 \cos(\lambda x) + C_2 \sin(\lambda x) \quad (9)$$

By applying the boundary conditions (8a) and (8b) into equation (9) yields:

$$X_n(x) = C_n \sin(n \pi x) \quad (10)$$

where $X_0(x) = 0$, $\lambda_n = n \pi$ and $n = 1, 2, 3, \dots$.

T-Problem Solver

An integration of the T -problem yields:

$$T_n(t) = B_n e^{-(n\pi)^2 t} \quad (11)$$

where $n = 1, 2, 3, \dots$.

Fundamental Solution

By substituting equations (10) and (11) into equation (4), it is possible to write the fundamental solution as:

$$u_n(x,t) = X_n(x)T_n(t)$$

$$u_n(x,t) = A_n \sin(n \pi x) e^{-(n\pi)^2 t}$$

where $u_0(x,t) = A_0$ and $n = 1, 2, 3, \dots$.

It is therefore possible to write the general solution of the one-dimensional homogeneous heat equation with the Neumann boundary condition as:

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(n \pi x) e^{-(n\pi)^2 t} \quad (12)$$

Specific Solution

When the equation (12) is applied to the initial condition (3), the equation (12) becomes:

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) = f(x) \tag{13}$$

where

$$A_n = \frac{2}{L} \int_0^L f(x) X(x) dx$$

$$A_n = 2 \int_0^1 x \sin(n\pi x) dx = \frac{2}{n\pi} [1 - (-1)^n] \tag{14}$$

where $n = 1, 2, 3, \dots$.

The complete solution of the one-dimensional homogeneous heat equation (1) can therefore be written as:

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{n\pi} [1 - (-1)^n] \sin(n\pi x) e^{-(n\pi)^2 t} \tag{15}$$

Results and discussions

The analytical solution of various boundary conditions on Dirichlet, Neumann and mixed boundary conditions is shown in Table 1, where the initial condition remains the same as $u(x,0) = x, 0 < x < 1$. The findings results show that, although the same initial conditions were used, there are different solutions.

Table 1. Analytical solution of different types of boundary conditions.

No.	Boundary condition	Solution
1	Dirichlet: $u(0,t) = 0, \quad t > 0$ $u(1,t) = 0, \quad t > 0$	$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{n\pi} [1 - (-1)^n] \sin(n\pi x) e^{-(n\pi)^2 t}$
2	Neumann: $u_x(0,t) = 0, \quad t > 0$ $u_x(1,t) = 0, \quad t > 0$	$u(x,t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2 \pi^2} [(-1)^n - 1] \cos(n\pi x) e^{-(n\pi)^2 t}$
3	Right end insulated: $u(0,t) = 0, \quad t > 0$ $u_x(1,t) = 0, \quad t > 0$	$u(x,t) = \sum_{n=1}^{\infty} \frac{8}{(2n-1)^2 \pi^2} (-1)^{n+1} \sin\left(\frac{(2n-1)\pi x}{2}\right) e^{-\left(\frac{(2n-1)\pi}{2}\right)^2 t}$

4 Left end insulated:

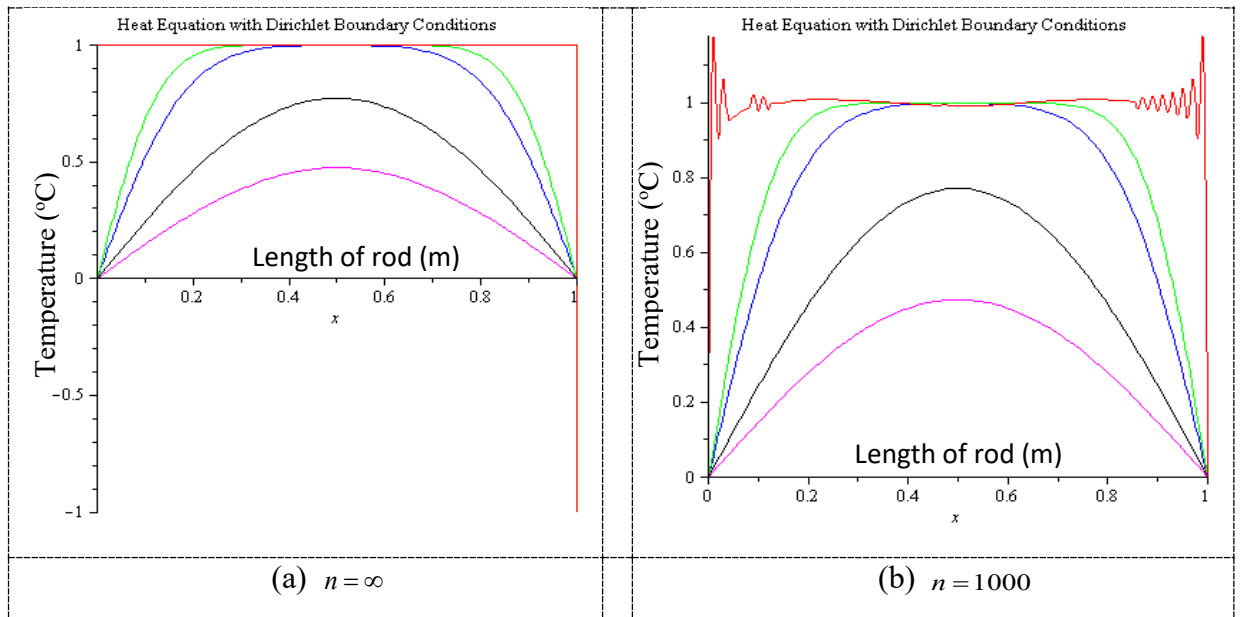
$$u_x(0,t) = 0, \quad t > 0$$

$$u(1,t) = 0, \quad t > 0$$

$$u(x,t) = \frac{1}{2} + \sum_{n=1}^{\infty} \left[-\frac{4}{(2n-1)\pi} + \frac{8}{(2n-1)^2 \pi^2} \right] \cos\left(\frac{(2n-1)\pi x}{2}\right) e^{-\left(\frac{(2n-1)\pi}{2}\right)^2 t}$$

Figure 2 show the heat profile along the rod at $t > 0$ for Dirichlet boundary conditions. The times are varying at 0.000s, 0.005s, 0.010s, 0.050s, 0.100s and 1.000s, at different number of iteration, $1 \leq n \leq \infty$. At time $t = 0.000$ s, the results of temperature profile are valid when $1 \leq n \leq 2$. After $n > 3$, the temperature starts to move up and down and the temperature start to drop extremely when $n = \infty$ because of the length of the rod, $L = 1$ m. The temperature at the left end and right end of the rod remain $0^\circ C$. The temperature start to increase when $x > 0$ and start to reduce after half length of the rod. The maximum value of temperature is $1.27^\circ C$.

Figure 3 show the heat profile along the rod at $t > 0$ for Neumann boundary conditions. The times are varying at 0.000s, 0.005s, 0.010s, 0.050s, 0.100s and 1.000s, at different number of iteration, $1 \leq n \leq \infty$. The results show the same pattern of temperature profile for all number of iteration $1 \leq n \leq \infty$. Since, the rod has insulated for both right and left end, the temperature do not spread from the rod. At the end of the rod, the temperature has move backward and this cause the highest value of temperature, $0.99^\circ C$. For Figure 3 (a), the temperature profile only shows the half cycle of the temperature distribution. Thus, the length of the rod need to be increased at $L = 2$ m to fulfil the complete cycle of temperature distribution.



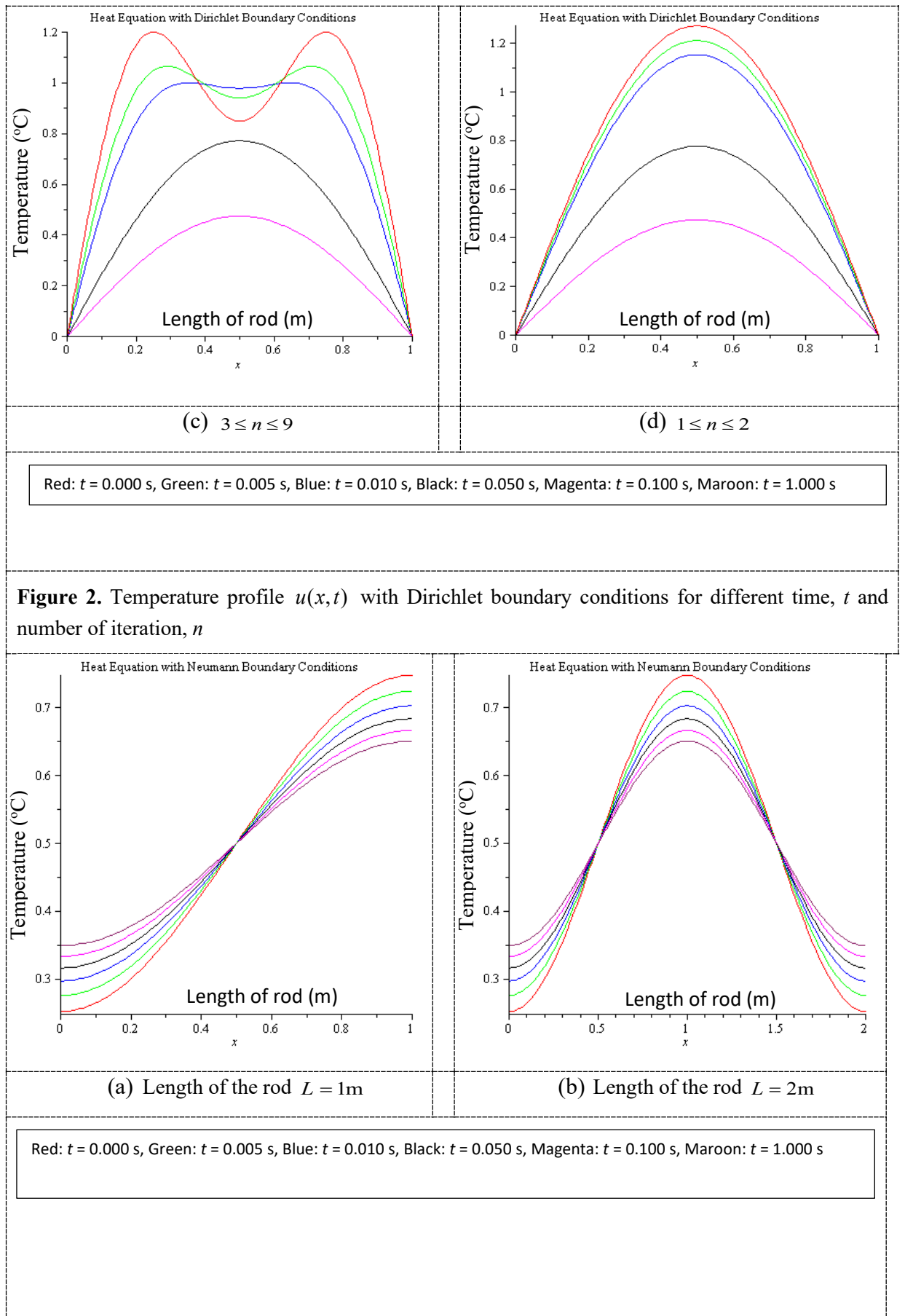


Figure 3. Temperature profile $u(x,t)$ with Neumann boundary conditions for different time, t and number of iteration, n

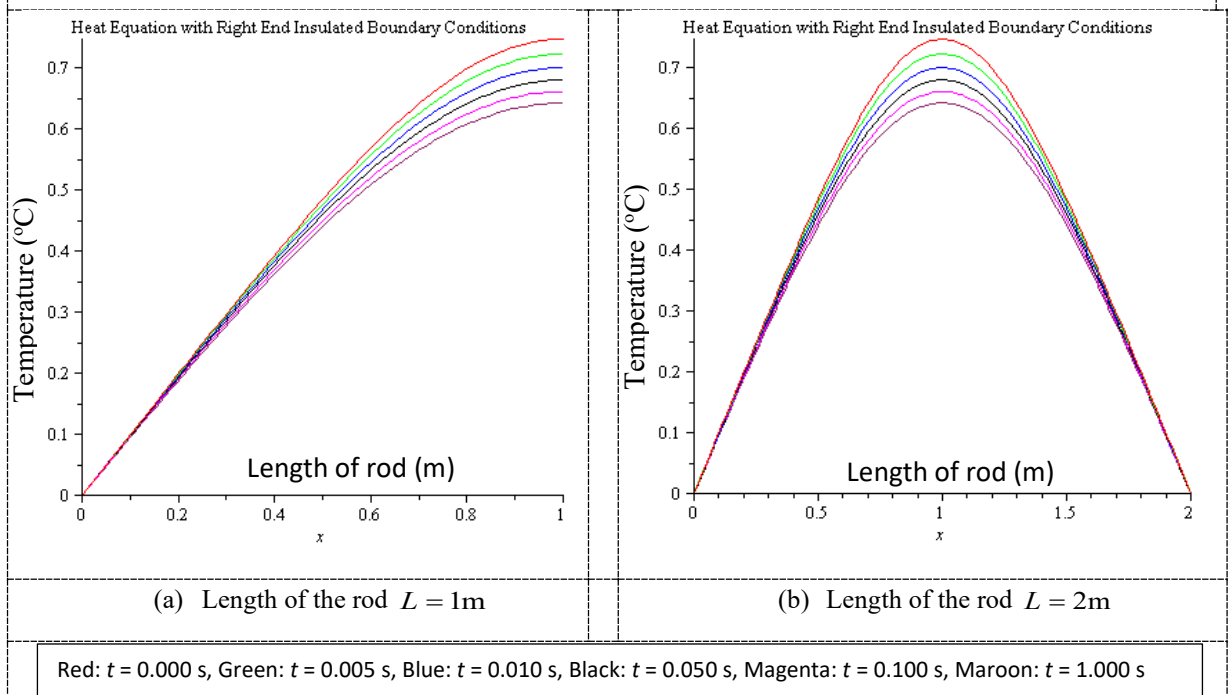


Figure 4. Temperature profile $u(x,t)$ with right end insulated boundary conditions for different time, t and number of iteration, n

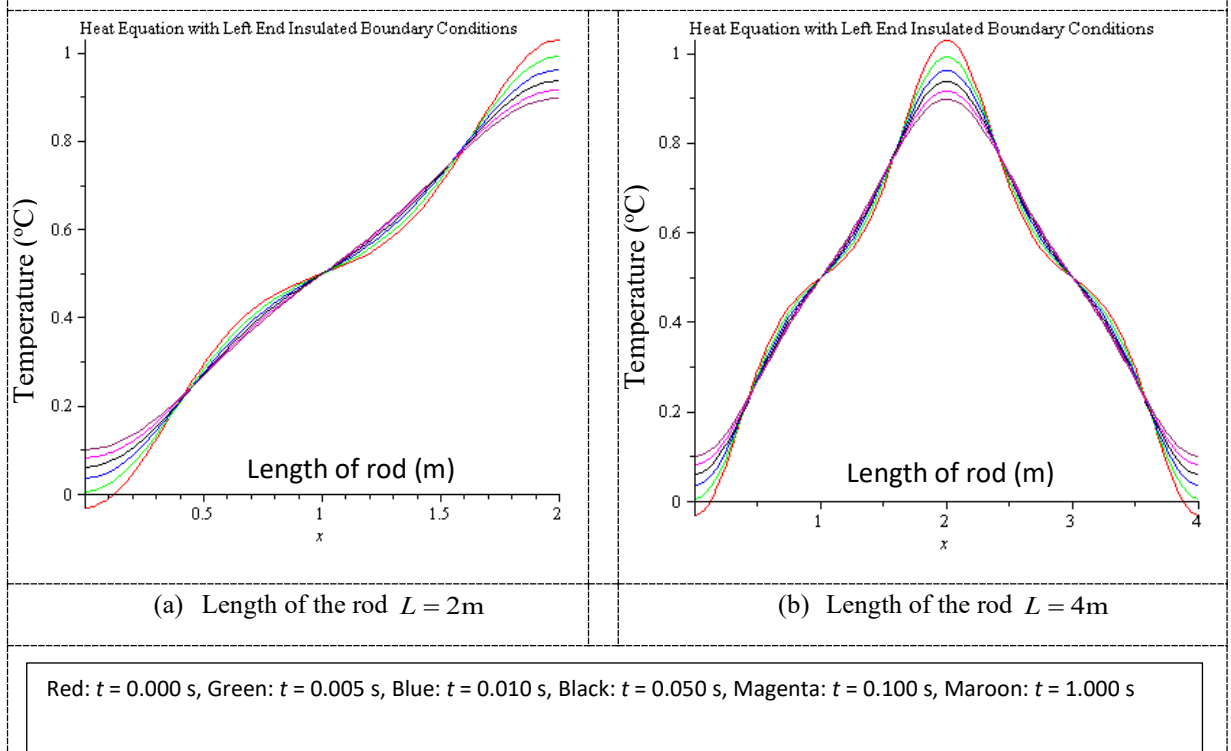


Figure 5. Temperature profile $u(x,t)$ with left end insulated boundary conditions for different time, t and number of iteration, n

Figure 4 illustrates the heat profile along the rod at $t > 0$ for right end insulated boundary conditions. The times are varying at 0.000s, 0.005s, 0.010s, 0.050s, 0.100s and 1.000s, at different number of iteration, $1 \leq n \leq \infty$. The results show the same pattern of temperature profile for all number of iteration $1 \leq n \leq \infty$. For Figure 4 (a), the temperature profile only shows the half cycle of the temperature distribution at the length of the rod $L = 1\text{m}$. Thus, the length of the rod need to be increased at $L = 2\text{m}$ to fulfil the complete cycle of temperature distribution. The highest temperature will occur at the centre of the rod $L = 1\text{m}$ which is 0.99°C and start reduced to zero until the end of the rod.

Figure 5 illustrates the heat profile along the rod at $t > 0$ for left end insulated boundary conditions. The times are varying at 0.000s, 0.005s, 0.010s, 0.050s, 0.100s and 1.000s, at different number of iteration, $1 \leq n \leq \infty$. The results show the same pattern of temperature profile for all number of iteration $1 \leq n \leq \infty$. For Figure 5 (a), the temperature profile only shows the half cycle of the temperature distribution at the length of the rod $L = 2\text{m}$. Thus, the length of the rod need to be increased at $L = 4\text{m}$ to fulfil the complete cycle of temperature distribution. The highest temperature will occur at the centre of the rod $L = 2\text{m}$ which is 1.97°C and start reduced to zero until the end of the rod.

Conclusions

For obtaining the exact solution of a partial differential equation, an analytical solution is necessary. The necessary boundaries and initial conditions are needed to solve these partial differential equations. Not only the equation, but also the boundary conditions depend on the general solution. In particular, if paired with varying sets of boundary conditions, these partial differential equations will have different general solutions. The homogenous one-dimensional heat equation will be analytically solved in the current analysis using the variable separation process. Our primary aim is to determine the heat equation's flow characteristics with boundary conditions of various types. To validate our objective, the heat equation will be solved based on Dirichlet, Neumann and mixed boundary conditions. The results are based on various boundary conditions, but the initial value remains the same. The times are varying at 0.000s, 0.005s, 0.010s, 0.050s, 0.100s and 1.000s, at different number of iteration, $1 \leq n \leq \infty$. At time $t = 0.000\text{s}$, the results of temperature profile for Dirichlet boundary conditions are valid when $1 \leq n \leq 2$. After $n > 3$, the temperature starts to move up and down and the temperature start to drop extremely when $n = \infty$ because of the length of the rod, $L = 1\text{m}$. The temperature at the left end and right end of the rod remain 0°C . The temperature start to increase when $x > 0$ and start to reduce after half length of the rod. However, there are the same pattern of temperature profile for all number of iteration $1 \leq n \leq \infty$. for Neumann boundary conditions. For right and left end insulated, the temperature profile only shows the half cycle of the temperature distribution at the length of the rod $L = 1\text{m}$ and $L = 2\text{m}$, respectively. Thus, the length of the rod need to be increased at $L = 2\text{m}$ and $L = 4\text{m}$ to fulfil the complete cycle of temperature distribution.

Conflict of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

- [1] Agyeman E and Folson D 2013 Algorithm analysis of numerical solutions to the heat equation *Int. J. of Comp. Appl.* **79** pp 975-8887
- [2] Mamun A A, Ali M S and Miah M M 2018 A study on an analytical solution 1d heat equation of a parabolic partial differential equation and implement in computer programming *Int. J. of Sci. and Eng. Res.* **9(9)** pp 2229 – 5518
- [3] Biala T A and Jator S N 2015 A boundary value approach for solving three dimensional elliptic and hyperbolic partial differential equations *SpringerPlus* **4(588)**

- [4] Papanikos G and Gousidou-Koutita M C 2015 A computational study with finite element method and finite difference method for 2D elliptic partial differential equations *Appl. Math.* pp 1-21
- [5] Subani N, Jamaluddin F, Mohamed M A H and Badrolhisam A D H 2020 Analytical solution of homogeneous one-dimensional heat equation with Neumann boundary conditions *J. of Phys.: Conf. Ser.* **1551 (2020)** pp 1-13
- [6] Abarbanel S, Ditkowski A and Gustafsson B 2000 On error bounds of finite difference approximations to partial differential equations – Temporal behavior and rate of convergence *J. of Sci. Comp.* **15** pp. 79-116
- [7] Islam M A, Hossain S M K and Rashid A 2018 Numerical solution of one-dimensional heat equation by Crank Nicolson method *Int. Conf. on Mechanical, Industrial and Energy Engineering* pp 1-4
- [8] Mebrate B 2015 Numerical solution of a one dimensional heat equation with Dirichlet boundary conditions *American. J. of Appl. Math.* **3(6)** pp 305-311
- [9] Roknujjaman M and Asaduzzaman M 2018 On the solution procedure of partial differential equation (PDE) with the methods of lines (MOL) using Crank-Nicholson method (CNM) *American. J. of Appl. Math.* **6(1)** pp 1-7
- [10] Tveito A and Winther R 1998 *Introduction to Partial Differential Equations: A Computational Approach* (New York: Springer-Verlag)
- [11] Crank J 1975 *The Mathematics of Diffusion* 2nd Ed. (London: Oxford University Press, Ely House)
- [12] Javed S 2012 *Thermal Modelling and Evaluation of Borehole Heat Transfer* (Sweden: Chalmers University of Technology)
- [13] Sabaiean M, Nadgaran H and Mousave L 2008 Analytical solution of the heat equation in a longitudinally pumped cubic solid-state laser *Opt. Soc. of America.* **47(13)** pp 2317-2325
- [14] Aidoo A and Wilson M 2015 A review of wavelets solution to stochastic heat equation with random inputs *Appl. Math.* **6** pp 2226-2239
- [15] As-Suyuti J 2003 *Tafsir Jalalain versi Terjemahan Melayu oleh Bahrin Abu Bakar, Sinar Baru Algensindo* (Bandung: Indonesia)
- [16] Al-Mahalli J 2003 *Tafsir Jalalain versi Terjemahan Melayu oleh Bahrin Abu Bakar, Sinar Baru Algensindo* (Bandung: Indonesia)
- [17] Al-Mahalli J and As-Suyuti J 2007 *Introduction to Tafsir Al-Jalalayn. Royal Aal al-Bayt Institute for Islamic Thought* (Amman: Jordan)
- [18] Emenogu G and Oko N 2015 Numerical solution of parabolic initial-boundary value problem with Crank-Nicolson's finite difference equations *IOSR J. of Math.* **11(4)** pp 16-19
- [19] Gerald W R 2011 *Finite-Difference Approximations to the Heat Equation* (Portland: Portland State University)
- [20] Jalil N F A 2011 *Solving One Dimensional Heat Equation and Groundwater Flow Modeling Using Finite Difference Method* (Malaysia: Universiti Teknologi Malaysia)
- [21] Zana S R Z 2014 *Numerical Solution of Diffusion Equation in One Dimension* (North Cyprus: Eastern Mediterranean University)
- [22] Dabral V, Kapoor S and Dhawan S 2011 Numerical simulation of one-dimensional heat equation: B-Spline finite element method *Indian J. of Comp. Sci. and Eng.* **2(2)**
- [23] Susan C, Brenner L and Ridgway S 2008 *The Mathematical Theory of Finite Element Methods*, 3rd Ed. (Spain: Springer)