CONFERENCE PROCEEDING

Comparison of Homogeneous and Non-Homogeneous One-dimensional Heat Equation

Muhammad Arif Hannan Mohamed, Muhammad Aniq Qayyum Mohamad Sukry, Faizzuddin Jamaluddin, Ahmad Danial Hidayatullah Badrolhisam, Norazlina Subani[1*]

Kolej GENIUS Insan, Universiti Sains Islam Malaysia, Bandar Baru Nilai, 71800, Nilai, Negeri Sembilan, Malaysia.
* norazlina.subani@usim.edu.my

Abstract

Heat equation is a partial differential equation that contains derivatives for two or more independent variables of an unknown function. There are heat equations that are homogenous and non-homogeneous. The non-homogeneous heat equation can be defined as there are source term in the partial differential equations, while there is no source term for the homogeneous heat equation. To solve the non-homogeneous, the heat equation need to be homogeneous by measure the displacement of the temperature of the heat from the equilibrium temperature. To obtain the exact solution of partial differential equation, an analytical solution is required. The suitable boundary and initial conditions are required to solve these partial differential equations. In the present research, the homogeneous and non-homogeneous one-dimensional heat equation will be solved analytically by using superposition principle (non-homogeneous) and separation of variables method (homogeneous). Our main objective is to compare the flow characteristics of heat equation on homogeneous and non-homogeneous heat equation. The heat equation will be solved based on Dirichlet boundary conditions to verify our objective. The finding results have been compared on the temperature profile for homogeneous and non-homogeneous heat equations. The temperature profile on the non-homogeneous Dirichlet boundary conditions show that it can be changed the profile and remain the same as temperature profile on homogeneous Dirichlet boundary condition when there is no source term.

Keywords: *GENIUS INAQ, heat equation, non-homogeneous one-dimensional, Dirichlet boundary conditions, analytical solution*

INTRODUCTION

A partial differential equation is an equation that involves derivatives of an unknown function with respect to two or more independent variables. It is possible to classify the partial differential equation into three form which are parabolic (Agyeman & Folson, 2003; Mamun *et al.,* 2018), hyperbolic and elliptic (Biala & Jator, 2015; Papanikos & Gousidou-Koutita, 2015). A parabolic partial differential equation that describes a large family of scientific problems, such as ocean acoustic propagation and heat diffusion. Hyperbolic partial differential equation that describes the wave transformation and vibrations of an elastic string, while elliptic partial differential equation describing the Laplace equation. The heat equation can be classified as non-homogeneous and homogeneous. Non-homogeneous heat equation can be defined as there are source term in the partial differential equations, while the homogeneous heat equation there are no source term.

According to Subani *et al*. (2020), a given partial differential equation can be solved by using numerical solution and analytical solution. However, to ensure that the numerical solution is valid (Abarbanel *et al*., 2000; Islam *et al*., 2018; Mebrate, 2015; Roknujjaman & Asaduzzaman, 2018), it is important to understand the general theory of partial differential equations. In order to obtain the exact solution of partial differential equation, an analytical solution is therefore required.

The suitable boundary and initial conditions are required to solve these partial differential equations. Not only the equation, but also the boundary conditions are depending on the general solution. In other words, when combined with various sets of boundary conditions, these partial differential equations would have distinct general solutions. However, they only consider the heat equation with Neumann boundary condition, and they found that the heat temperature quickly converges to zero compared to the long rod.

The heat equation propagates energy, which is highly non-physical, at an infinite speed. However, for all classical physics and engineering applications, the validity of the heat equation as a model of temperature evolution is still extremely strong. Temperature fluctuations are one of the main consequences of heat transfer, where the heating process raises the temperature while the cooling process reduces the temperature (Tveito & Winther, 1998). This assumes that there is no step shift in this process and that no work is performed on or by the system (Crank, 1975). Javed (2012) studies on sources of dry or moist heat. Hot water bottles, radiant heat and electric pads provide dry applications. Moist heat is known to be more penetrating than dry heat, although this is more due to the slower loss of heat from water-soaked materials than dry ones.

Sabaeian *et al.* (2008) claimed that the in Islamic perspective, that the role of temperature distribution is important in the measurement, simulation, and prediction of thermal effects. Temperature is specific to heat capacity, or the quantity of energy needed to adjust the substance's temperature. The measuring heat changes due to physical or chemical changes (Aidoo & Wilson, 2015).

According to Al-Mahalli (2003) and As-Suyuti (2003), the verse in *Quran* (*Surah Yassin*: 80) tells us about the development of the fire from green trees. In other words, by using green plants, fire can be made. In that life, the friction of two surface objects generates fire (Al-Mahalli & As-Suyuti, 2007). In this research, the heat velocity would be calculated from one zone of high to low heat. The rate of heat velocity is dependent upon the degree of friction velocity between the two objects.

In the present research, the homogeneous and non-homogeneous one-dimensional heat equation will be solved analytically by using superposition principle (non-homogeneous) and separation of variables method (homogeneous). Our main objective is to compare the flow characteristics of heat equation on homogeneous and non-homogeneous heat equation. The heat equation will be solved based on Dirichlet boundary conditions to verify our objective. The finding results have been compared on the temperature profile for homogeneous and non-homogeneous heat equations.

MATHEMATICAL FORMULATION

The mathematical models are used to describe the one-dimensional non-homogeneous heat boundary value problems with Dirichlet boundary conditions are presented below. The heat equation is used to determine the change in the function of temperature, u over time, t. The simplified diagram of a physical model of the heat equation problem is shown in Figure 1.

$$u(x,0) = f(x)$$

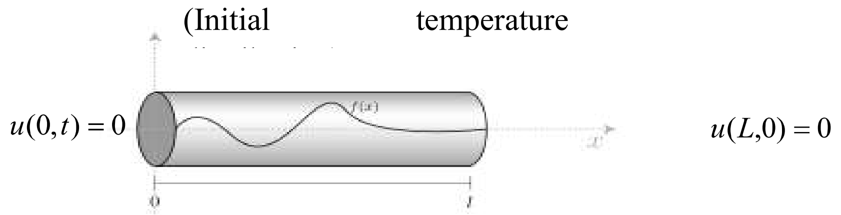


Figure 1. Simplified diagram of physical model of heat equation problem with Dirichlet boundary condition

Boundary Value Problem

The partial differential equation of one-dimensional non-homogeneous heat conduction equation is given by:

$$u_t(x,t) = u_{xx}(x,t) + F(x), \quad 0 < x < 1, \quad t > 0 \tag{1}$$

where u is defined as heat temperature, x is space, t is time and $F(x) = 1$ is non-homogenous or other resource term.

Boundary and Initial Conditions

There are three types of boundary conditions involving this problem which are Dirichlet, Neumann and mixed of the boundary conditions (left end insulated and right end insulated).

The Dirichlet boundary conditions at the initial point, $x = 0$ and at the end point $x = 1$ are given by:

$$u(0,t) = 100, \quad t > 0 \text{ and } u(1,t) = 100, \quad t > 0 \tag{2}$$

The initial conditions at $t = 0$ is:

$$u(x,0) = x, \quad 0 < x < 1 \tag{3}$$

These mathematical models of Eqs. 1-3 described the heat conduction in a one dimensional uniform rod of length one unit with internal heat sources (non-homogeneous part), thermal diffusivity one, perfect lateral insulation and initial condition, x when $0 < x < 1$.

ANALYTICAL SOLUTION

Using Crank-Nicolson (Emenogu & Oko, 2015), finite different method (Gerald, 2011; Jalil, 2011; Zana, 2014) or finite element method (Dabral *et al*., 2011; Susan *et al*., 2008), most of the heat equation will be solved numerically. However, in order to obtain the exact solution of the partial differential equation, an analytical solution is needed. To solve analytically the partial differential equation (1), the method of superposition principle is used to be homogeneous part of the equations. The superposition principle can have applied to any linear system, such as heat equation.

Superposition can be defined by two simpler properties which are additivity and homogeneity. In this research, the non-homogeneous heat equation will be classified as homogeneous property. Thus, the method of Separation of Variables (SOV) will be to solve the homogeneous heat equation. The single partial differential equation can be separated into two ordinary differential equations, where there is only one independent variable for each equation.

To solve the non-homogeneous heat equation, the partial differential (1) need to be homogeneous. Thus, the Eq. 1 can be written as:

$$u(x,t) = v(x,t) + w(x)$$

(4)

where

$$u_t(x,t) = v_t(x,t) \text{ and } u_{xx}(x,t) = v_{xx}(x,t) + w''(x)$$

(5)

Now, substituting Eq. 5 into Eq. 1 yields:

$$v_t(x,t) = v_{xx}(x,t) + w''(x) + 1$$

(6)

To solve the Eq. 6, the non-homogeneous part of the heat partial differential equation 6 need to be homogeneous. By integrating the non-homogeneous part of Eq. 6 yields:

$$w(x) = -\frac{x^2}{2} + c_1 x + c_2$$

(7)

By using boundary conditions (2), the new boundary conditions become:

$$u(0,t) = v(0,t) + w(0) = 100 \text{, where } v(0,t) = 0 \text{ and } w(0,t) = 100$$

(8a)

$$u(1,t) = v(1,t) + w(1) = 100 \text{, where } v(1,t) = 0 \text{ and } w(1,t) = 100$$

(8b)

Substituting Eqs. 8a-8b into Eq. 7 becomes:

$$w(x) = -\frac{x^2}{2} + \frac{x}{2} + 100$$

(9)

Then, the homogenous of partial differential equations now can be written as:

$$\begin{aligned}
&v_t(x,t) = v_{xx}(x,t), \quad 0 < x < 1, \quad t > 0 \\
&v(0,t) = 0, \quad t > 0 \\
&v(1,t) = 0, \quad t > 0 \\
&v(1,0) = x - w(x), \quad 0 < x < 1
\end{aligned}$$

(10)

Eq. 10 describe the homogeneous heat equation. Separation of Variables (SOV) is used to analytically address a partial differential Eq. 10. The single partial differential equation can be separated into two ordinary differential equations, where there is only one independent variable for each equation (Subani *et al*. 2020). Then, the general solution of homogeneous one-dimensional heat equation with Dirichlet boundary condition can be written as:

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)\, e^{-(n\pi)^2 t}$$

(11)

By applying the initial condition (10) into equation (11) yields:

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)\, e^{-(n\pi)^2 t} - \frac{x^2}{2} + \frac{x}{2} + 100$$

(12)

where

$$A_n = 2\int_0^1 \left[\frac{x^2}{2} - \frac{x}{2} - 100 \right] \sin(n\pi x)\, dx$$

$$A_n = \frac{2}{n\pi} \left[99(-1)^n + \frac{(-1)^n - 1}{(n\pi)^2} - 100 \right]$$

(13)

where $n = 1, 2, 3, \cdots$.

RESULTS AND DISCUSSION

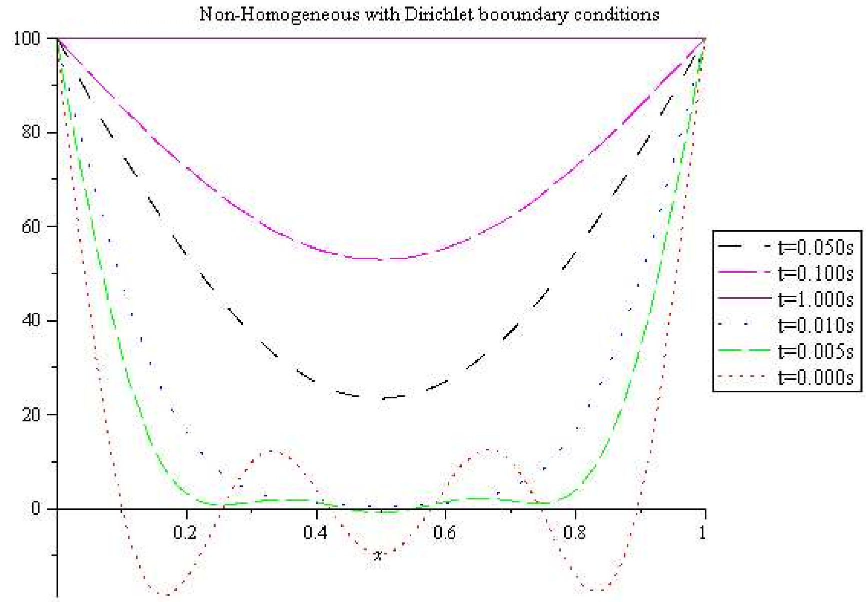
Table 1 shows the analytical solution homogeneous and non-homogeneous heat equation for Dirichlet boundary condition. and the initial condition remain the same as $u(x,0) = x, \quad 0 < x < 1$. The results show that there have different solutions although there used the same initial conditions.

Table 1. Analytical solution for homogeneous and non-homogeneous heat equation for Dirichlet boundary condition

No.	Types of Equation	Boundary condition	Solution
1	Homogeneous	$u(0,t) = 0, \quad t > 0$ $u(1,t) = 0, \quad t > 0$	$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \left[1 - (-1)^n\right] \sin(n\pi x)\, e^{-(n\pi)^2 t}$

2	Non-homogeneous	$u(0,t)=100, \quad t>0$ $u(1,t)=100, \quad t>0$	$u(x,t)=\sum_{n-1}^{\infty}\frac{2}{n\pi}\left[99(-1)^n+\frac{(-1)^n-1}{(n\pi)^2}-100\right]$ $\left[\sin(n\pi x)e^{-(n\pi)^2 t}\right]-\frac{x^2}{2}+\frac{x}{2}+100$

Figure 2 show the heat profile along the rod at $t>0$ for Dirichlet boundary conditions for homogeneous and non-homogeneous heat equation. The times are varying at 0.000s, 0.005s, 0.010s, 0.050s, 0.100s and 1.000s, at number of iteration, ∞ and the length of the rod, L=1m. The results in Figure 2 are based on the solution of heat equation in Table 1. The temperature profiles are remaining stable for both homogeneous and non-homogeneous heat equation when time t=0.010s, t=0.050s and t =0.100s. At the time t =1.000s, the temperature for non-homogeneous and homogeneous heat equation kept at their boundary conditions, $100.0^\circ C$ and $0^\circ C$, respectively. Based on the results in Figure 2, the temperature profile on the non-homogeneous Dirichlet boundary conditions show that it can be changed the profile and remain the same as temperature profile on homogeneous Dirichlet boundary condition when there is no source term.



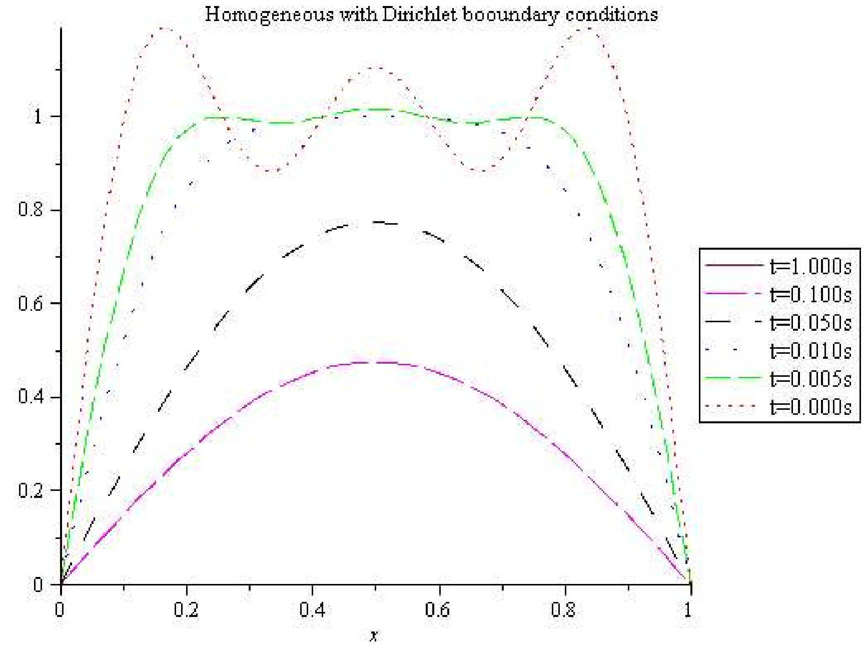


Figure 2. Temperature profile $u(x,t)$ with Dirichlet boundary conditions for homogeneous and non-homogeneous heat equation at different time, *t*

CONCLUSION

The heat equation solution depends not only on the equation, but also on the conditions or initial conditions of the boundary. The suitable boundary and initial conditions are required to solve these partial differential equations. When combined with various sets of boundary conditions or initial conditions, these partial differential equations would have distinct general solutions. To obtain the exact solution of partial differential equation, an analytical solution is required. The non-homogeneous and homogeneous one-dimensional heat equation will be analytically solved in the current research using the superposition principle (non-homogeneous) and separation of variables method (homogeneous). Our main objective is to compare the flow characteristics of heat equation on homogeneous and non-homogeneous heat equation. The heat equation will be solved based on Dirichlet boundary conditions to verify our objective. The finding results have been compared on the temperature profile for homogeneous and non-homogeneous heat equations. The temperature profiles are remaining stable for both homogeneous and non-homogeneous heat equation when time *t* =0.010s, *t* =0.050s and *t* =0.100s. At the time *t* =1.000s, the temperature for non-homogeneous and homogeneous heat equation kept at their boundary conditions, $100.0^{\circ}C$ and $0.0^{\circ}C$, respectively. The temperature profile on the non-homogeneous Dirichlet boundary conditions show that it can be changed the profile and remain the same as temperature profile on homogeneous Dirichlet boundary condition when there is no source term.

Length of rod (m)

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests regarding the publication of this paper.

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REFERENCES

Abarbanel, S., Ditkowski, A. & Gustafsson, B. 2000. "On Error Bounds of Finite Difference Approximations to Partial Differential Equations – Temporal Behavior and Rate of Convergence". *Journal of Science Computer*, Vol. 15. p. 79-116.

Agyeman, E. & Folson, D. 2013. "Algorithm Analysis of Numerical Solutions to the Heat Equation". *International Journal of Computational Application,* Vol. 79. p. 975-8887.

Aidoo, A. & Wilson, M. 2015. "A Review of Wavelets Solution to Stochastic Heat Equation with Random Inputs". *Applied Mathematics,* Vol. 6. p. 2226-2239.

Al-Mahalli, J. & As-Suyuti, J. 2007. Introduction to Tafsir Al-Jalalayn. Royal Al-Bayt Institute for Islamic Though. Amman: Jordan.

Al-Mahalli, J. 2003. Tafsir Jalalain versi Terjemahan Melayu oleh Bahrun Abu Bakar, Sinar Baru Algensindo. Bandung: Indonesia.

As-Suyuti, J. 2003. Tafsir Jalalain versi Terjemahan Melayu oleh Bahrun Abu Bakar, Sinar Baru Algensindo. Bandung: Indonesia.

Biala, T. A. & Jator, S. N. 2015. "A Boundary Value Approach for Solving Three Dimensional Elliptic and Hyperbolic Partial Differential Equations". *SpringerPlus*, Vol. 4. (588).

Crank, J. 1975. The Mathematics of Diffusion. 2nd Edition. London: Oxford University Press, Ely House.

Dabral, V., Kapoor, S. & Dhawan, S. 2011. "Numerical Simulation of One-Dimensional Heat Equation: B-Spline Finite Element Method". *Indian Journal of Computer Science and Engineering,* Vol. 2. (2).

Emenogu, G. & Oko, N. 2015. "Numerical Solution of Parabolic Initial-Boundary Value Problem with Crank-Nicolson's Finite Difference Equations". *IOSR Journal of Mathematics,* Vol. 11. (4). p. 16-19.

Gerald, W. R. 2011. Finite-Difference Approximations to the Heat Equation. Portland: Portland State University.

Islam, M. A., Hossain, S. M. K. & Rashid, A. 2018. "Numerical Solution of One-Dimensional Heat Equation by Crank Nicolson Method". *International Conference on Mechanical, Industrial and Energy Engineering,* p. 1-4.

Jalil, N. F. A. 2011. Solving One Dimensional Heat Equation and Groundwater Flow Modeling Using Finite Difference Method. Malaysia: Universiti Teknologi Malaysia.

Javed, S. 2012. Thermal Modelling and Evaluation of Borehole Heat Transfer. Sweeden: Chalmers University of Technology.

Mamun, A. A., Ali, M. S. & Miah, M. M. 2018. "A Study on an Analytical Solution 1D Heat Equation of a Parabolic Partial Differential Equation and Implement in Computer Programming". *International Journal of Science and Engineering. Research,* Vol. 9. (9). p. 2229 – 5518.

Mebrate, B. 2015. "Numerical Solution of a One Dimensional Heat Equation with Dirichlet Boundary Conditions". *American Journal of Applied Mathematics,* Vol. 3. (6). p. 305-311.

Papanikos, G. & Gousidou-Koutita, M. C. 2015. "A Computational Study with Finite Element Method and Finite Difference Method for 2D Elliptic Partial Differential Equations". *Applied Mathematics*, p. 1-21.

Roknujjaman, M. & Asaduzzaman, M. 2018. "On the Solution Procedure of Partial Differential Equation (PDE) with the Methods of Lines (MOL) using Crank-Nicholson Method (CNM)". *American Journal of Applied Mathematics,* Vol. 6. (1). p. 1-7.

Sabaeian, M., Nadgaran, H. & Mousave, L. 2008. "Analytical Solution of the Heat Equation in a Longitudinally Pumped Cubic Solid-State Laser". *Journal of the Optical Socity of America,* Vol. 47. (13). p. 2317-2325.

Subani, N., Jamaluddin, F., Mohamed, M. A. H. & Badrolhisam, A. D. H. 2020. "Analytical Solution of Homogeneous One-Dimensional Heat Equation with Neumann Boundary Conditions". *Journal of Physics: Conference Series,* Vol. 1551. (2020). p. 1-13.

Susan, C., Brenner, L. & Ridgway, S. 2008. The Mathematical Theory of Finite Element Methods. 3rd Edition. Spain: Springer.

Tveito, A. & Winther, R. 1998. Introduction to Partial Differential Equations: A Computational Approach. New York: Springer-Verlag.

Zana, S. R. Z. 2014. Numerical Solution of Diffusion Equation in One Dimension. North Cyprus: Eastern Mediterranean University.