Ain Shams Engineering Journal 9 (2018) 3359–3363

Contents lists available at ScienceDirect

Ain Shams Engineering Journal

journal homepage: www.sciencedirect.com

Homotopy perturbation method for the hypersingular integral equations of the first kind

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ARTICLE INFO

Article history:
Available online 16 November 2018

MSC:
65R20
45E05

Keywords:
Homotopy perturbation method
Hypersingular integral
Numerical method
Convergence

ABSTRACT

Simple and efficient convex homotopy perturbation method (HPM) is presented to obtain an approximate solution of hyper-singular integral equations of the first kind. Convergence and error estimate of HPM are obtained. Three numerical examples were provided to verify the effectiveness of the HPM. Comparisons with reproducing kernel method (Chen et al., 2011) for the same number of iteration is also presented. Numerical examples reveal that the convergence of HPM can still be achieved for some problems even if the condition of convergence of HPM is not satisfied.

1. Introduction

Hypersingular integral equations (HSIEs) comes from a variety of mixed boundary value problems in mathematical physics such as water wave scattering [1] and radiation problems involving thin submerged plates [2,3] and fracture mechanics [4]. Chen et al. [5] have solved HSIE using the improvement of reproducing kernel method. Convergence of Galerkin and collocation method were discussed in Golberg [6] to obtain the approximate solution of HSIEs. Spline collocations method has also been used in Boykov et al. [7,8] to solve the linear and nonlinear HSIEs of the first and second kind respectively. Moreover, the authors Eshkuvatov et al. [9,10] have used projection method with Chebyshev polynomials to solve singular and hypersingular integral equations, respectively. Nik Long and Eshkuvatov [11] have used the complex variable function method to formulate the multiple curved crack problems into hypersingular integral equations. These hypersingular integral equations are solved numerically for the unknown function, which are later used to find the stress intensity factor. Furthermore, Antangana and Bildik [12] have solved fractional Volterra integral equations of second kind by Simpson 3/8 rule method and in Atangana [13] new derivative fractional order is used to solve nonlinear Fisher's reaction-diffusion equation.

In the resent decades homotopy perturbation method (HPM) has been used to solve different types of singular, hyper-singular integral equation and integro-differential equations. He J.H. proposed HPM in 1999 and since then it used for a wide range of problems [14–16]. Particularly, HPM is applied for solving nonlinear ordinary differential equations (ODEs) [17], one-phase inverse Stefan problem [18], linear and nonlinear integral equations [19], the integro-differential equations [20,21], fractional partial differential equations [22] and the Volterra-Fredholm integral equations [23]. The convergence of the decomposition method has been discussed in Buldik [24] for Fredholm and Volterra integral equation. Hetmaniok et al. [25] have applied HPM for Volterra-Fredholm integral equations and proved the convergence of the HPM.

Consider HSIE of the first kind

$$\frac{1}{\pi}\,⨎_{-1}^{1}\frac{u(t)}{(x-t)^2}\,dt+\frac{\alpha}{\pi}\int_{-1}^{1}K(x,t)\,u(t)\,dt=f(x),\quad -1<x<1, \tag{1}$$

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Peer review under responsibility of Ain Shams University.

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https://doi.org/10.1016/j.asej.2017.04.010
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where $f(x)$ is given square integrable function, $K(x,t)$ is the square integrable kernel and $u(x)$ is the unknown function to be determined.

Let us search the bounded solution of Eq. (1) of the form

$$u(x) = \sqrt{1-x^2}\, g(x), \tag{2}$$

then Eq. (1) can be written as

$$\frac{1}{\pi}\fint_{-1}^{1} \frac{\sqrt{1-t^2}\, g(t)}{(x-t)^2}\, dt + \frac{\alpha}{\pi}\int_{-1}^{1} K(x,t)\sqrt{1-t^2}\, g(t) dt = f(x), \tag{3}$$

Rewrite Eq.(3) in operator form

$$Hg + \alpha Kg = f, \tag{4}$$

where

$$Hg(x) = \frac{1}{\pi}\fint_{-1}^{1} \frac{\sqrt{1-t^2}\, g(t)}{(x-t)^2}\, dt, \quad Kg(x) = \frac{1}{\pi}\int_{-1}^{1} K(x,t)\sqrt{1-t^2}\, g(t)\, dt, \quad f = f(x). \tag{5}$$

The structure of the paper is arranged as follows: in Section 2, we provide some theoretical aspects in Hilbert space. In Section 3, derivation of the formula and convergence of the HPM for Eq. (4) are given. Numerical examples are presented in Section 4. Finally, conclusion is given in Section 5.

2. Preliminaries

Summarizing the well known properties concerning the operator H defined by (5). Let

$$U_n(\theta) = \frac{\sin[(n+1)\theta]}{\sin\theta}, \quad \theta = \cos^{-1} x, \quad n = 0,1,2,\ldots, \tag{6}$$

denotes the Chebyshev polynomials of the second kind, and

$$\phi_n = \sqrt{\frac{2}{\pi}} U_n, \tag{7}$$

where ϕ_n are normalized so that

$$\int_{-1}^{1} \sqrt{1-t^2}\, \phi_n^2\, \mathrm{d}t = 1. \tag{8}$$

It is well known [6] that

$$H\phi_n = -(n+1)\phi_n, \quad n = 0,1,2,\ldots, \tag{9}$$

Let $L(\rho)$ denote the space of square integrable real valued function with respect to $\rho(x) = \sqrt{1-x^2}$. The inner product on $L(\rho)$ is given by

$$\langle g, v\rangle_\rho = \int_{-1}^{1} \rho(t) g(t)\, v(t)\, dt, \tag{10}$$

and $\|g\|_\rho = \sqrt{\langle g, v\rangle_\rho}$ denotes the norm.

The set $\{\phi_k\}_{k=0}^{\infty}$ is a complete orthonormal basis for $L(\rho)$ i.e. for any $g \in L(\rho)$ then

$$g = \sum_{k=0}^{\infty} \langle g, \phi_k\rangle_\rho \phi_k, \tag{11}$$

where the series (11) converges in $L(\rho)$. In addition, the norm of g satisfies the Parseval's equality

$$\|g\|_\rho^2 = \sum_{k=0}^{\infty} \langle g, \phi_k\rangle_\rho^2. \tag{12}$$

We need the subspace of $L(\rho)$ which is consisting of all g such that

$$\sum_{k=0}^{\infty} (k+1)^2 \langle g, \phi_k\rangle_\rho^2 < \infty. \tag{13}$$

All functions satisfying (13) are denoted by $L_1(\rho)$ and it can be made into Hilbert space if the inner product of $g \in L_1(\rho)$ and $v \in L_1(\rho)$ are defined by

$$\langle g, v\rangle_1 = \sum_{k=0}^{\infty} (k+1)^2 \langle g, \phi_k\rangle_\rho \langle v, \phi_k\rangle_\rho. \tag{14}$$

The norm of $g \in L_1(\rho)$ is then given by

$$\|g\|_1^2 = \sum_{k=0}^{\infty} (k+1)^2 \langle g, \phi_k\rangle_\rho^2. \tag{15}$$

We extend H as a bounded operator from $L_1(\rho)$ to $L(\rho)$ by defining

$$Hg = \sum_{k=0}^{\infty} \langle g, \phi_k\rangle_\rho H\phi_k = \sum_{k=0}^{\infty} -\langle g, \phi_k\rangle_\rho (k+1)\phi_k, \tag{16}$$

and observe that

$$\|Hg\|_\rho^2 = \sum_{k=0}^{\infty} (k+1)^2 \langle g, \phi_k\rangle_\rho^2 = \|g\|_1^2. \tag{17}$$

It can be easily shown [6] that $H^{-1} : L(\rho) \to L_1(\rho)$ is given by

$$H^{-1} g = \sum_{k=0}^{\infty} \left(-\frac{\langle g, \phi_k\rangle_\rho}{k+1}\right)\phi_k, \tag{18}$$

so that H is unitary. Consequently, H^{-1} exists.

Lemma 1. *The norm of operator $H^{-1} : L_1(\rho) \to L(\rho)$ is $\|H^{-1}\| = 1$ and*

$$\|H^{-1} g\|^2 = \sum_{k=0}^{\infty} \left(\frac{\langle g, \phi_k\rangle}{k+1}\right)^2. \tag{19}$$

Proof. Eq. (19) can easily be obtained by using Eq. (18). For the norm of H^{-1} we assume that $H^{-1}g = v$. On the other hand

$$\langle v, \phi_k\rangle = \langle H^{-1} g, \phi_k\rangle = -\frac{\langle g, \phi_k\rangle}{k+1}. \tag{20}$$

Since $v \in L_1(\rho)$ and due to (20) we have

$$\|v\|_1^2 = \sum_{k=0}^{\infty} (k+1)^2 (\langle v, \phi_k\rangle)^2 \tag{21}$$

$$= \sum_{k=0}^{\infty} (\langle g, \phi_k\rangle)^2 \tag{22}$$

$$= \|g\|_\rho^2. \tag{23}$$

Therefore,

$$\|H^{-1}\|_1 = \|g\|_\rho.$$

By the definition of norm operator yields

$$\|H^{-1}\| = \sup_{\substack{g\in L_1(\rho)\\ \|g\|_\rho \leqslant 1}} \|H^{-1} g\|_1 = \sup_{\|g\|_\rho < 1} \|g\|_\rho = 1. \quad \square \tag{24}$$

3. Derivation and convergence of the HPM

In this section, we present the application of HPM for solving HSIE of the first kind. Let the perturbation scheme in convex homotopy form

$$H^*(v,p) = (1-p)F(v) + p(Hv + \alpha Kv - f) \tag{25}$$

where $F(v)$ is a functional operator with initial solution g_0 and $p \in [0,1]$ is called homotopy parameter. When $H^*(v,0) = 0$, the solution of the operator equation is equivalent to the solution of a trivial problem $Hv(x) - g_0(x) = 0$. For $H^*(v,1) = 0$ leads to the solution of Eq. (4). $H^*(v,p)$ shows the curve is continuously traces from a starting point $H^*(v_0,0)$ to a solution $H^*(v,1)$. The solution of operator equation $H^*(v,p) = 0$ is searched in the form of power series

$$v(x) = \sum_{k=0}^{\infty} p^k v_k(x). \tag{26}$$

When $p \to 1$, Eq. (25) becomes

$$v(x) = \lim_{p\to 1} v(x) = v_0 + v_1 + \dots. \tag{27}$$

Substituting $F(v) = Hv - g_0$ and $H^*(v,p) = 0$ into Eq. (25), we obtain

$$Hv = g_0 + p(f - \alpha K v - g_0). \tag{28}$$

We assume that the series (26) has radius of convergence not smaller than 1 and that it is absolutely convergent. Applying series (26) into (28) yields

$$H\left(\sum_{k=0}^{\infty} p^k v_k(x)\right) = g_0 + p\left[f - \alpha K\left(\sum_{k=0}^{\infty} p^k v_k(x)\right) - g_0\right]. \tag{29}$$

Comparing the like power of parameter p in Eq. (29), leads to the following approximate solution

$$Hv_0 = g_0, \tag{30}$$

$$\begin{aligned} Hv_1 &= f - \alpha K v_0 - g_0, \\ Hv_2 &= -\alpha K v_1, \\ Hv_k &= -\alpha K v_{k-1}. \end{aligned} \tag{31}$$

Since H^{-1} exist, we have

$$\begin{aligned} v_0 &= H^{-1} g_0, \\ v_1 &= H^{-1}(f - \alpha K v_0 - g_0), \\ v_2 &= -\alpha H^{-1} K v_1, \\ v_k &= -\alpha H^{-1} K v_{k-1}, \end{aligned} \tag{32}$$

The convergence of HPM when $p = 1$ in Eq. (26) and error of *nth* order are discussed in following theorems.

Theorem 1. *Let g in be a smooth function, $f \in C[-1,1]$ and $K \in C([-1,1] \times [-1,1])$. If the following inequality*

$$|\alpha|\,\|K\| < 1 \tag{33}$$

is satisfied and initial guess $v_0(t)$ is chosen as a continuous function with $Hv_0 = g_0$ for any $t \in [1,1]$, then series (26)–(32) converge to the exact solution g for any $p \in [-1,1]$ in the norm $\|\cdot\|_1$.

Proof. Let $g(x)$ be a smooth function and the functions $f(x), K(x,t)$ are continuous in $[-1,1]$ and set the relations (32) in the norm $\|\cdot\|_1$ as follows

$$\begin{aligned} \|v_0\|_1 &\leqslant \|H^{-1}\|\|g_0\|_1, \\ \|v_1\|_1 &\leqslant \|H^{-1}\|(\|f\|_1 + |\alpha|\,\|K\|_1\,\|v_0\|_1 + \|g_0\|_1), \\ \|v_k\|_1 &\leqslant |\alpha|\,\|H^{-1}\|\,\|K\|_1\,\|v_{k-1}\|_1. \end{aligned} \tag{34}$$

Since $H^{-1} : L(\rho) \to L_1(\rho)$ we have $v_k = \alpha H^{-1} K v_{k-1} \in L_1(\rho)$ and $\|H^{-1}\| = 1$ gives

$$\|v_k\|_1 \leqslant \|\alpha K\|_1\,\|v_{k-1}\|_1, \quad k = 1,2,\dots. \tag{35}$$

Consequently

$$\|v_k\|_1 \leqslant \|\alpha K\|_1^{k-1}\,\|v_1\|_1, \quad k = 1,2,\dots. \tag{36}$$

From (26) we obtain

$$\begin{aligned} \|v\| &\leqslant \sum_{k=0}^{\infty} |p^k|\,\|v_k\|_1 = \|v_0\|_1 + \|v_1\|_1 + \sum_{k=2}^{\infty} \|v_k\|_1, \\ &\leqslant \|v_0\|_1 + \|v_1\|_1 + \sum_{k=2}^{\infty} |\alpha K\|_1^{k-1}\,\|v_1\|_1. \end{aligned} \tag{37}$$

The last series in (37) is the convergent series possessing the common ratio $|\alpha K\|_1 < 1$. Hence, $v(x)$ is convergent in the norm $\|\cdot\|_1$. □

The first $n+1$ terms of series (26) is called approximate solution in the form

$$\hat{v}_n(x) = \sum_{k=0}^{n} v_k(x). \tag{38}$$

Estimation of the solution $\hat{v}_n(x)$ for Eq. (1) is based on the following theorem.

Theorem 2. *Let $\|\alpha K\| < 1$, then the error of nth-order approximate solution in (38) is estimated as*

$$E_n \leqslant \frac{\|\alpha K\|_1^n}{1 - \|\alpha K\|_1}\,\|v_1\|_1, \tag{39}$$

where $E_n = \|g - \hat{v}_n\|_1$ and $\|K\|_1$ is the norm of integral operator.

Proof. Since the series v in (26) converges to the exact solution g we can consider the difference $\|g - \hat{v}\| = \|v - \hat{v}\|$.

$$\begin{aligned} \|v - \hat{v}_n\|_1 &= \left\| \sum_{k=n+1}^{\infty} v_k \right\|_1, \\ &\leqslant \sum_{k=n+1}^{\infty} \|v_k\|_1, \\ &\leqslant \sum_{k=n+1}^{\infty} \|\alpha K\|_1^{k-1}\,\|v_1\|_1, \\ &= \frac{\|\alpha K\|^n}{1 - \|\alpha K\|_1}\,\|v_1\|_1. \quad \square \end{aligned} \tag{40}$$

Remark 1. Since $\|\alpha K\|_1 < 1$, and $\|v_1\|$ is bounded, then

$$\frac{\|\alpha K\|_1^n}{1 - \|\alpha K\|_1}\,\|v_1\|_1 \to 0$$

as $n \to \infty$.

4. Numerical example

Example 1. Consider the HSIE of the form

$$\begin{aligned} &\frac{1}{\pi}\int_{-1}^{1} \frac{u(t)}{(x-t)^2}\,dt + \frac{1}{3}\int_{-1}^{1} \frac{2xt}{\pi}(9 + 32x^2t^2 - 16x^2 \\ &\quad - 16t^2)\,u(t)\,dt \\ &\quad = \frac{\sqrt{2}(10 + 10x - \sqrt{5} + 4\sqrt{5}x^2)}{5\sqrt{\pi}}, \end{aligned} \tag{41}$$

Table 1
Errors of the approximate solutions $\hat{v}_n(x)$ for Eq. (41).

n	$E_n = \|u - \hat{v}_n\|_1$ for Eq. (41)
4	5.2280203×10^{-8}
6	$9.0764241 \times 10^{-11}$
8	$1.5757681 \times 10^{-13}$
10	$2.735708454 \times 10^{-16}$

which has the exact solution $u_{exact}(x) = -\sqrt{1-x^2}\frac{\sqrt{2}}{345\sqrt{\pi}}(690 + 360x - 23\sqrt{5} + 92\sqrt{5}x^2)$.

Since the kernel $K(x,t)$ is continuous the inequality

$$\|\alpha K\| \approx 0.3992 < 1. \tag{42}$$

holds, hence the condition in Theorem 1 is satisfied. This example is solved by using HPM with initial guess $g_0 = \phi_0(x)$ for $n = \{4, 6, 8, 10\}$. Results of norm errors $\|\hat{v}_n - u_{exact}\|$ is shown in Table 1.

Example 2. Consider HSIE

$$\frac{1}{\pi}\int_{-1}^{1}\frac{u(t)}{(x-t)^2}dt + \frac{1}{\pi}\int_{-1}^{1}(2t + x^2)\ln(3-x)u(t)dt$$
$$= -\frac{11}{2}x + 20x^3 - 24x^5 + \frac{3}{8}\ln(3-x), \quad -1 < x < 1, \tag{43}$$

Exact solution is $u_{exact}(x) = \sqrt{1-x^2}(\frac{7}{4}x - 3x^3 + 4x^5)$.

Checking the conditions in Theorem 1, yields

$$\|\alpha K\| \approx 2 \nleq 1, \tag{44}$$

Conditions of Theorem 1 is not satisfied for Eq. (43). But we still tested Eq. (43) by using HPM. The approximate solutions is calculated with initial guess $g_0(x) = \phi_1(x)$ for $n = \{4, 6, 8, 10\}$.

Example 3. Chen and Zhou [5] has consider HSIE

$$\frac{1}{\pi}\int_{-1}^{1}\frac{u(t)}{(x-t)^2}dt + \frac{1}{\pi}\int_{-1}^{1}(t+x)u(t)dt$$
$$= \frac{1}{2}(1 - 6x^2) + \frac{1}{8}x, \quad -1 < x < 1, \tag{45}$$

Exact solution is $u_{exact}(x) = \sqrt{1-x^2}x^2$.

It can be easily shown that $\|\alpha K\| > 1$ for Eq. (45) which is not satisfied the condition of Theorem 1. Comparison results between HPM and reproducing kernel [5] are shown in Table 3. In approximate solutions the initial guess $g_0(x) = \phi_0(x)$.

5. Conclusion

In this paper, HPM is used and analyzed for solving HSIE of the first kind. Mainly, bounded solution of the problem is considered. Theorem 1 shows that HPM is convergent for HSIE if $|\alpha|\,\|K\| < 1$. In Example 1 all condition of Theorem 1 is satisfied therefore HPM converges to exact solution very fast. In Examples 2 and 3 corresponding (Tables 2–4), HPM is still convergent even though the condition of Theorem 1 is not satisfied. It shows that Theorem 1 has only necessary conditions, the inequalities $\|\alpha H\| > 1$ shows that the propose method either converge or diverge. Fortunately, Eqs. (43) and (45) converges when the number iteration increased. Thus, HPM is highly accurate and reliable method in solving hypersingular integral equation of the first kind.

Table 2
Errors of the approximate solutions $\hat{v}_n(x)$ for Eq. (43).

n	$E_n = \|u - \hat{v}_n\|_1$ for Eq. (41)
4	2.6576280×10^{-4}
6	2.3538932×10^{-6}
8	1.0788644×10^{-8}
10	$4.9836288 \times 10^{-11}$

Table 3
Errors of solutions for Eq. (45).

N	Chen and Zhou [5]	HPM
5	1.5×10^{-7}	1.2791560×10^{-4}
15	3.6×10^{-8}	$3.8121818 \times 10^{-12}$

Table 4
Comparisons between HPM and reproducing kernel method for Eq. (43).

n	$E_n = \|u - \hat{v}_n\|_1$ for Eq. (41)
4	1.0233248×10^{-3}
6	$3.19788989 \times 10^{-5}$
8	9.9934059×10^{-7}
10	3.1229394×10^{-8}

Acknowledgment

This work was supported by Universiti Sains Islam Malaysia (USIM), Short Project code PPP/USG-0216/FST/30/15316 and University Putra Malaysia (UPM) under Geran Putra, 2014, Project code GP-i/2014/9442300. Authors are grateful for sponsorship and financial support of the Research Management Center, USIM and UPM.

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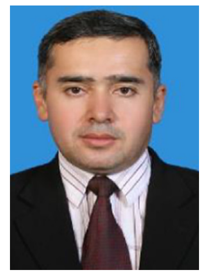

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