NUMERICAL ALGEBRA,
CONTROL AND OPTIMIZATION
Volume 7, Number 1, March 2017

doi:10.3934/naco.2017004

pp. 77–88

EFFECTIVE APPROXIMATION METHOD FOR SOLVING LINEAR FREDHOLM-VOLTERRA INTEGRAL EQUATIONS

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Abstract. An efficient approximate method for solving Fredholm-Volterra integral equations of the third kind is presented. As a basis functions truncated Legendre series is used for unknown function and Gauss-Legendre quadrature formula with collocation method are applied to reduce problem into linear algebraic equations. The existence and uniqueness solution of the integral equation of the 3rd kind are shown as well as rate of convergence is obtained. Illustrative examples revels that the proposed method is very efficient and accurate. Finally, comparison results with the previous work are also given.

1. **Introduction.** There are many books [1, 5, 9–11, 20] written on linear integral equations (IEs) and scientific papers on linear IEs [2–4, 6, 7, 12–15, 17–19], system of linear IEs [8, 16] and literatures cited therein.

Many problems in mathematical physics and engineering can be recast into Fredholm-Volterra integral equation of the form

$$a(s)\,x(s) = f(s) + \lambda_1 \int_D K_1(s,t)\,x(t)\,dt + \lambda_2 \int_B K_2(s,t)\,x(t)\,dt, \quad s \in D, \qquad (1)$$

2010 *Mathematics Subject Classification.* Primary: 45B05; Secondary: 45A05, 45L05.

Key words and phrases. Integral equations, Legendre polynomials, approximations, collocation method, Galerkin method, regular kernel.

The first author is supported by USIM grant PPP/GP/FST/30/14915.

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The reviewing process of the paper was handled by Wah June Leong as Guest Editor.

where $B \subseteq D$ is a closed bounded set in $\mathbb{R}^m$, $m \geq 1$. The kernels $K_i(s,t)$, $i = \{1,2\}$ are assumed to be square integrable on the prescribed region D and $a(s) \in C(D)$. For $f \neq 0$, and we seek x; this is inhomogeneous problem. For $f = 0$, Eq. (1) becomes eigenvalues problem, and we seek both the eigenvalues λ and eigenfunctions x.

Particularly, in 2007, Babolian and Fattahzadeh [2] proposed direct method for solving integral equations whose solutions are continuous or discontinuous by using Chebyshev wavelet basis in Galerkin equations. On the other hand for solving Volterra type integral equations operational matrix of integration together with Chebyshev wavelets is introduced and used it to reduce the problem to a system of algebraic equations.The numerical examples and the number of operations show the advantages of Chebyshev Wavelet Galerkin method to some other usual methods and usual Chebyshev basis. In 2009, Chakrabarti and Martha [6] solve a special class of Fredholm integral equations of the second kind. The unknown function is approximated by the linear combination of Bernstein polynomials of degree n and the least-squares method is used to solve the resulting over-determined system of equations. Several illustrative examples are examined in details. Maleknajad et. al. [13] developed expansion method for solving Eq. (1) on the interval $[-1,1]$. Legendre polynomials are chosen as the base functions. For the convergence of the method, they referred to the known classical theorems and provided examples which are good fit for certain type of the function $f(s)$ of Eq. (1). Elliot [7] proposed the expansion method for kernel $K(s,t)$ and the unknown function $x(s)$ of Eq. (1). Chebyshev truncated series was used to approximate the kernel and unknown functions. Error estimations and convergence of the method were also discussed. Mandal and Bhattacharya [17] utilized the expansion method for the solution function $x(s)$ in Eq. (1), in the form

$$x(s) = \sum_{i=1}^{n+1} c_{i-1} B_{i-1,n}(s),$$

where c_i are unknown constants and $B_{i-1,n}(s)$, $i = 1, ..., n+1$ are Bernstein polynomials of degree n defined on an interval $[a,b]$. The integral equations considered are Fredholm integral equations of second kind, and a hypersingular integral equation of second kind. The method is explained with illustrative examples. Also, the convergence of the method is established for each class of integral equations considered. In 2009, Nik Long et al. [19] solve infinite boundary integral equation (IBIE) of the second kind numerically. Galerkin method with Laguerre polynomial is applied to get the approximate solution. Numerical examples are given to show the validity of the method presented.

In this work we consider Fredholm-Volterra integral equations of the third kind

$$a(s)\,x(s) = f(s) + \lambda_1 \int_a^b K_1(s,t)\,x(t)\,dt + \lambda_2 \int_a^s K_2(s,t)\,x(t)\,dt, \tag{2}$$

where $f(s)$ is given continuous function on $[a,b]$ and kernels $K_i(s,t)$, $i = \{1,2\}$ are the square integrable in the domain $D = \{(s,t): \ a \leq s,t \leq b\}$, while λ_i, $i = \{1,2\}$ are the constant parameters and $x(s)$ is unknown function to be determined.

In the solution of Eq. (2), Nystrom type Gauss-Legendre quadrature formula (QF) together with Legendre truncated series are implemented. Collocation points are chosen as the roots of Legendre polynomials.

The manuscript is structured as follows. Section II describes Legendre polynomials, its properties and Gaus-Legendre quadrature formula. The derivation of the new proposed method is given in the Section III. The existence solution and exactness of the approximate method are discussed in Section IV. Section V deals with the numerical examples and comparisons with Chakrabarti and Martha [6], Melaknejad [13] as well as Mustafa [18].

2. **Preliminaries.**

2.1. **Legendre polynomials.** Recall Legendre polynomials $P_n\left(s\right)$

Definition 2.1. Legendre functions are the solutions to Legendre's differentiae equation

$$\frac{d}{ds}\left[\left(1-s^2\right)\frac{dy}{ds}\right]+n(n+1)y=0 \tag{3}$$

The solutions of Eq. (3) for $n=0,1,2,...$ (with the normalization P_n) form a polynomial sequence of orthogonal polynomial, called the Legendre polynomials denoted by $P_n\left(s\right)$. Each Legendre polynomials $P_n\left(s\right)$ is a nth-degree polynomials.

1. It may be expressed using Rodrigues formula

$$P_n\left(s\right)=\frac{1}{2^n n!}\frac{d^n}{ds^n}\left(s^2-1\right)^n,\ \ P_0\left(s\right)=1$$

2. or $P_n\left(s\right)$ can also be defined as the coefficients in a Taylor series expansion.

$$\frac{1}{\sqrt{1-2st+t^2}}=\sum_{n=0}^{\infty}P_n(s)t^4$$

3. or three term recurrence relations

$$\begin{array}{l} P_0\left(s\right)=1,\ \ P_1\left(s\right)=s, \\ P_{n+1}\left(s\right)=\frac{2n+1}{n+1}sP_n\left(s\right)-\frac{n}{n+1}P_{n-1}\left(s\right),\ \ n\geq 1 \end{array}$$

2.2. **Properties.**

1. Legendre polynomials (LP) $\left\{P_n\left(s\right)\right\}_{n=0}^{\infty}$ form to complete orthogonal system in $L_2\left[-1,1\right]$, i.e. any piecewise function $f\left(s\right)\in L_2\left[-1,1\right]$ can be expressed in terms of Legendre Polynomials

$$\sum_{n=0}^{\infty}c_nP_n\left(s\right)=f\left(s\right)$$

if $f\left(s\right)$ is discontinuous at c of the first kind then

$$\frac{f(c^-)+f(c^+)}{2}=\sum_{n=0}^{\infty}c_nP_n\left(s\right)$$

2. Legendre polynomials are even or odd function depending on its degree i.e.

$$P_n\left(-s\right)=\left(-1\right)^nP_n\left(s\right)$$

3. Derivative of Legendre polynomial is

$$P_n^{'}\left(-s\right)=\left(-1\right)^{n+1}P_n^{'}\left(s\right)$$

4. Recurrence relations: [Legendre polynomials at one point can be expressed by neighboring Legendre polynomials at the same point]
   - $(2n+1)P_n\left(s\right)=P_{n+1}^{'}\left(s\right)-P_{n-1}^{'}\left(s\right)$
   - $\left(s^2-1\right)P_n^{'}\left(s\right)=nsP_n\left(s\right)-nP_{n-1}\left(s\right)$

- $P_{n-1}^{'}(s) = sP_{n}^{'}(s) - nP_{n}(s)$
- $P_{n+1}^{'}(s) = sP_{n}^{'}(s) + (n+1)P_{n}(s)$

5. The roots of Legendre polynomials $P_n(s)$ are given by

$$s_i = \left(1 - \frac{1}{8n^2} + \frac{1}{8n^3}\right)\cos\left(\frac{4i-1}{4n+2}\right)\pi, \quad i = 0, 1, ...n$$

6. Legendre polynomials are orthogonal polynomials

$$\langle P_m, P_n\rangle = \int_{-1}^{1} P_m(s)P_n(s)ds = \begin{cases} 0, & m \neq n, \\ \|P_n\|_2^2, & m = n. \end{cases}$$

where $\|P_n\|_2$denotes $L2$ norm and

$$\|P_n\|_2 = \sqrt{\frac{2}{2n+1}}, \quad n = 0, 1, 2, ...$$

2.3. **Gauss-Legendre quadrature formula:**

It is known that Legendre polynomials $P_n(s)$ are the orthogonal polynomials on $[-1,1]$ with weights $w(x) = 1$, therefore Gauss-Legendre quadrature formula (QF) of the form

$$\int_{-1}^{1} f(\tau)\,dt = \sum_{i=1}^{n+1} w_i f(\tau_i) + R_{n+1}(f), \tag{4}$$

where

$$w_i = \frac{2}{(1-\tau_i^2)\left[P_{n+1}'(\tau_i)\right]^2}, \qquad \sum_{i=1}^{n+1} w_i = 2, \tag{5}$$

is exact for the polynomial of degree $2n+1$ if weights w_i are defined by (5) and collocation points τ_i are chosen as the roots of Legendre polynomials $P_{n+1}(s)$ i.e.

$$P_{n+1}(\tau_i) \equiv 0, \quad i = 1, 2, ..., n+1. \tag{6}$$

Error term of QF (4) (Kythe and Schaferkotter [11]) is

$$R_n(f) = \frac{2^{2n+1}(n!)^4}{(2n+1)\left[(2n)!\right]^3} f^{(2n)}(\xi), \quad -1 < \xi < 1.$$

Extending the Gauss-Legendre QF (4) to the kernel integral on the $[a,b]$ yields

$$Q_1(s) = \int_a^b K(s,t)\,x(t)\,dt = \frac{b-a}{2}\sum_{k=1}^{n+1} W_k(s)\,x(t_k) + R_n(x), \tag{7}$$

$$Q_2(s) = \int_a^s K(s,t)\,x(t)\,dt = \frac{s-a}{2}\sum_{k=1}^{n+1} W_k(s)\,x(t_k) + R_n(x), \tag{8}$$

where parameter s can be any values in $[a,b]$ and

$$W_k(s) = K(s,t_k)\,w_k, \quad t_k = \frac{b-a}{2}\tau_k + \frac{b+a}{2},$$

here τ_k is defined by Eq. (6).

Eqs. (7) and (8) are crucial for the rest of analysis.

3. **Description of the Method.** Let us rewrite Eq. (2) in the form

$$x\left(s\right)=\frac{f\left(s\right)}{a\left(s\right)}+\lambda_1\int_a^b\frac{K_1\left(s,t\right)}{a\left(s\right)}x\left(t\right)dt+\lambda_2\int_a^s\frac{K_2\left(s,t\right)}{a\left(s\right)}x\left(t\right)dt, \tag{9}$$

and search solution $x\left(s\right)$ as follows

$$x\left(s\right)\approx x_n\left(s\right)=\sum_{j=0}^{n}c_jP_j\left(s\right), \quad s=\frac{b-a}{2}\tau+\frac{b+a}{2},\ \tau\in\left[-1,1\right], \tag{10}$$

Substituting (10) into (9) and implementing Gauss-Legendre QF (7) and (8), yields

$$\sum_{j=0}^{n}c_j\left[P_j(s)-\sum_{k=1}^{n+1}\left(\lambda_1\frac{b-a}{2}W_{1k}\left(s\right)+\lambda_2\frac{s-a}{2}W_{2k}\left(s\right)\right)P_j\left(t_k\right)\right]=\frac{f\left(s\right)}{a\left(s\right)}, \tag{11}$$

where

$$W_{1k}\left(s\right)=\frac{K_1\left(s,t_k\right)}{a\left(s\right)}w_k, \qquad W_{2k}\left(s\right)=\frac{K_2\left(s,t_k\right)}{a\left(s\right)}w_k, \tag{12}$$

and w_k are defined by (5).

The unknown coefficients c_j in Eq. (11) are determined by choosing the collocation points

$$s=s_i=\frac{b-a}{2}\tau_i+\frac{b+a}{2},\ \ i=1,2,...,n+1,$$

where $\tau_i\in(-1,1)$ are defined by Eq. (6). Collocation method for Eq. (11) leads to the system of algebraic equations

$$AC=f \tag{13}$$

where

$$C=\left(c_0,c_1,...,c_n\right)^T.\text{and } f=\left(\frac{f_1\left(s_1\right)}{a\left(s_1\right)},\frac{f_1\left(s_2\right)}{a\left(s_2\right)},...,\frac{f_1\left(s_{n+1}\right)}{a\left(s_{n+1}\right)}\right)^T$$

$$\psi_j\left(s_i\right)=P_j\left(s_i\right)-\sum_{k=1}^{n+1}\left(\lambda_1\frac{b-a}{2}W_{1k}\left(s_i\right)+\lambda_2\frac{s_i-a}{2}W_{2k}\left(s_i\right)\right)P_j\left(t_k\right),$$
$$j=0,...,n,\quad i=1,...,n+1,$$

$$A=\begin{pmatrix}\psi_0\left(s_1\right) & \psi_1\left(s_1\right) & \cdots & \psi_n\left(s_1\right)\\ \psi_0\left(s_2\right) & \psi_1\left(s_2\right) & \cdots & \psi_n\left(s_2\right)\\ \vdots & \vdots & \ddots & \vdots\\ \psi_0\left(s_{n+1}\right) & \psi_1\left(s_{n+1}\right) & \cdots & \psi_n\left(s_{n+1}\right)\end{pmatrix}$$

From Eqs. (10) and (13) we obtain the approximate solution of Eq. (2). The system of equations (13) has a unique solution if matrix A is nonsingular.

4. **Existence and Uniqueness.** Let us rewrite Eq. (9) in the operator form

$$x+Kx=g, \tag{14}$$

where $Kx=\lambda_1K_1x+\lambda_2K_2$

$$K_ix=\lambda_i\int_a^b\frac{K_i(s,t)}{a(s)}x(t)dt,\ \ i=\{1,2\}$$

$$g(s)=\frac{f(s)}{a(s)},\ \ a(s)\neq0,\ \ s\in D$$

In Atkinson [1] shown that if D is closed bounded set in $\mathbb{R}^m$, $m \geq 1$ and K is defined as

$$Kx(s) = \int_D K(s,t)x(t)dt, \;\; s \in D, \;\; x \in C(D)$$

then $K : C(D) \to C(D)$ is both bounded and compact in $C(D)$ with $\|.\|_\infty$

Lemma 4.1. *(Atkinson [1]) Let $K \in L[X,Y]$ and $L \in L[Y,Z]$, and let K or L (or both) be compact. Then K is compact on X to Z. Here $L[X,Y]$ denotes linear operator from X to Y.*

Theorem 4.2. *(Fredholm alternative) Let X be a Banach space, and let $K : X \to X$ be compact. Then the equation $(\lambda I - K)x = y$, $\lambda \neq 0$ has a unique solution $x \in X$ if and only if the homogeneous equation $(\lambda I - K)z = 0$ has only the trivial solution $z = 0$. In such a case, the operator $\lambda I - K : X \underset{onto}{\overset{1-1}{\to}} X$ has a bounded inverse $(\lambda I - K)^{-1}$.*

Consider the set of $n+1$ collocation points s_j which are the zeros of P_{n+1}

$$s_j = \left(1 - \frac{1}{8n^2}\right) \cos\left(\frac{4j-1}{4n+2}\right)\pi, \;\; j = 0, 1, 2, ..., n.$$

Consider that function $e_0, e_1, ..., e_n$ in X such that

$$e_j(s_k) = \delta_{jk},$$

define the projection operators $\pi_n X$ from X into X, by

$$\pi_n x(s) = \sum_{j=0}^{n} e_j(s) s(s_j),$$

where $e_j(s) = \sqrt{\frac{2j+1}{2}} p_j(s)$ is a normalized Legendre sequance. We consider the sequance of finite rank orthogonal projections (π_n) defined by

$$\pi_n X := \sum_{j=0}^{n} \langle x, e_j \rangle. \tag{15}$$

Let H:=$L^2([a,b], C)$ be Hilbert space and consider the approximate problem of finding $\pi_n X \in H$ such that

$$\pi_n x + K\pi_n x = g. \tag{16}$$

Let operator K is approximated by $K_n s \cdot t$. $\|K_n - K\| \to 0, n \to \infty$. Applying operator $\pi_n X$ for$x$ and $K_n x$ for Kx we arrive at

$$\pi_n x + K_n \pi_n x = g, \tag{17}$$

where it is shown in Atkinson [1] that

$$\|\pi_n x - x\|_\infty \to 0, \;\; for\, all x \in X$$
$$\|K_n x - Kx\|_\infty \to 0, \;\; for\, all x \in X$$

Let

$$C = \sup_{n \geq N} \left\|(I + K_n)^{-1}\right\| \tag{18}$$

It is shown in Atkinson [1] that inverse operator $(I + K_n)^{-1}$ exists and is uniformly bounded for n large enough. That is the constant c is finite. Main Theorem.

Theorem 4.3. *Assume that* $g \in X = C[a,b]$ *and* $\|K_n x - Kx\| \underset{n\to\infty}{\rightarrow} 0$. *The following estimate holds for* n *large enough.*

$$\|x_n - x\|_\infty \le C\|K_n x - Kx\|_\infty$$

Proof. From (15) we have

$$x = (I+K)^{-1} g.$$

From (17)

$$x_n = (I+K_n)^{-1} g.$$

Then

$$\begin{aligned} x_n - x &= [(I+K_n)^{-1} - (I+K)^{-1}]g \\ &= (I+K_n)^{-1}[(I+K) - (I+K_n)](I+K_n)^{-1} g \\ &= (I+K_n)^{-1}[(K-K_n]g. \end{aligned}$$

Since C defined by (18) is bounded we have

$$\begin{aligned} \|x_n - x\|_\infty &= \|(I+K_n)^{-1}\|_\infty \|Kx - K_n x\|_\infty \\ &\le C\|Kx - K_n x\|_\infty. \end{aligned}$$

The theorem is proved. □

5. **Results and Discussions.** Let us introduce the error terms:

- $Qe_n(s) = |x(s) - x_n(s)|$ is the difference between the exact $x(s)$ and the approximate solution $x_n(s)$ for Eq. (2).
- $Ce_n(s) = |x(s) - Cx_n(s)|$ is the error term of Chakrabarti [6],
- $Me_n(s) = |x(s) - Mx_n(s)|$ is the error term of Melaknejad [13],
- $Mue_n(s) = |x(s) - Mux_n(s)|$ is the error term of Mustafa [18].

Example 1: Consider Eq. (2) with $\lambda_1 = -1$, $\lambda_2 = 0$, $a = 0$, $b = 1$ and

$$\begin{aligned} &a(s) = 1,\ \ f(s) = x^2, \\ &K_1(s,t) = -(s^2+t^2),\ \ K_2(s,t) = 0. \end{aligned}$$

The exact solution of Eq. (2) is $x(s) = \dfrac{9}{11} + \dfrac{30}{11}s^2$.
We show that proposed method is exact. Let us rewrite the equation in the form

$$x(s) = s^2 + \int_0^1 \left(s^2+t^2\right) x(t)\, dt, \tag{19}$$

and choose $n = 2$, then

$$x_2(s) = \sum_{i=0}^{2} c_i P_i(s). \tag{20}$$

Substitute Eq. (15) into Eq. (14) to get

$$\sum_{i=0}^{2} c_i \left[P_i(s) - \int_0^1 \left(s^2+t^2\right) P_i(t)\, dt\right] = s^2. \tag{21}$$

By applying Gauss-Legendre (QF) and choosing roots as

$$P_3(s_i) = 0,\ \ i = 1,2,3.$$

We determine the unknowns $c_i,\ i = \{0, 1, 2\}$

$$c_0 = \frac{19}{11},\ \ c_1 = 0,\ \ c_2 = \frac{20}{11} \tag{22}$$

Substitute (17) into (15) yields

$$x_2(s) = x(s) = \frac{9}{11} + \frac{30}{11}s^2,$$

which is identical to the exact solution. For other values of "n" the errors $Qe_n(s)$ is shown in Table 1.

s	$Qe_n(s)(11)$		
	n=5	n=11	n=20
1	3.000e-19	1.000e-19	4.000e-19
0.8	2.000e-19	1.000e-19	4.000e-19
0.6	1.000e-19	0.000e-0	3.000e-19
0.4	0.000e-0	0.000e-0	1.000e-19
0.2	5.000e-20	7.000e-20	7.000e-20
0.1	3.000e-20	1.000e-19	7.000e-20
0	6.000e-20	1.100e-19	6.000e-20

TABLE 1. The error term $Qe_n(s) = |x(s) - Qx_n(s)|$ for Example 1

Table 1, reveals that the proposed method is exact for Example 1 and it is shown analytically as well.

Examples 2 (Chakrabarti [6]): Let $\lambda_1 = -1$, $\lambda_2 = 0$, $a = 0$,$b = 1$ and

$$a(s) = 1, f(s) = 1 + s,$$
$$K_1(s,t) = -\left(\sqrt{s} + \sqrt{t}\right), \quad K_2(s,t) = 0.$$

The exact solution of Eq. (2) is $x(s) = -\frac{129}{70} + s - \frac{141}{35}\sqrt{s}$. For different values of "n" the error of proposed method $Qe_n(s)$ is presented in Table 2.

s	$Qe_n(s)(11)$		
	$n = 5$	$n = 10$	$n = 20$
1.0	3.571e-3	2.676e-3	3.999e-4
0.9	9.534e-3	1.518e-3	2.031e-4
0.7	1.445e-3	1.501e-3	1.800e-4
0.5	1.014e-2	8.541e-4	1.368e-4
0.3	3.406e-3	5.103e-4	5.103e-4
0.1	2.674e-2	3.278e-3	3.673e-4
0.0	4.155e-1	2.331e-1	1.244e-1

TABLE 2. The error term $Qe_n(s) = |x(s) - Qx_n(s)|$ for Example 2

Comparisons with Chakrabarti [6] is summarized in Table 3.

Since the proposed method is not exact for the type of solution $x(s) = -\frac{129}{70} + s - \frac{141}{35}\sqrt{s}$ and derivative of the Fredolm kernel $K_1(s,t) = -\left(\sqrt{s} + \sqrt{t}\right)$ is not differentiable at the point $\{0\}$ the error term $Qe_n(s)$ **is not** drastically decreases, fortunately it is comparable with the method proposed by Chakrabarti [6].

s	$n=2$	
	$Qe_n(s)$(11)	$Ce_n(s)$ [6]
1.00	9.943e-2	9.500e-2
0.75	1.874e-3	6.380e-2
0.50	3.183e-2	3.690e-2
0.25	9.565e-2	8.200e-2
0.00	7.925e-1	6.927e-1

TABLE 3. Error comparison between $Qe_n(s)$ and $Ce_n(s)$ for Example 2

Examples 3(Maleknejad [13]): Consider Eq. (2) where $\lambda_1 = 1$, $\lambda_2 = 0$, $a = -1$, $b = 1$ and

$$a(s) = 1, f(s) = -\frac{3}{3\pi^2} + \cos(2\pi s) + \frac{6s}{4\pi^2 - 9s^2}\sin(3s)$$

$$K_1(s,t) = \sin(2\pi s + \pi t) + \cos(3st) + \frac{st^2}{3}, \quad K_2(s,t) = 0,$$

with exact solution is $x(s) = \cos(2\pi s)$.

For Example 3 we have given two Tables 4 and Table 5 to show the validity of Maleknejad [13] and proposed method (11).

s	$n=5$		$n=7$		$n=9$	
	$Qe_n(s)$(11)	$Me_n(s)$ [13]	$Qe_n(s)$(11)	$Me_n(s)$ [13]	$Qe_n(s)$(11)	$Me_n(s)$ [13]
0.999	6.935e-1	1.829e-1	2.908e-1	1.109e-1	4.768e-2	1.562e-2
0.753	2.829e-1	1.026e0	6.962e-2	2.700e-1	1.581e-2	1.445e-2
0.352	4.374e-1	1.009e0	1.355e-1	1.502e-1	1.387e-2	3.929e-2
0.001	6.416e-1	3.025e-1	1.412e-1	1.401e-1	1.750e-2	1.152e-2
-0.001	6.411e-1	3.058e-1	1.412e-1	1.386e-1	1.750e-2	1.123e-2
-0.352	5.173e-1	4.409e-1	1.370e-1	4.024e-2	1.388e-2	2.596e-2
-0.753	4.027e-1	1.165e-1	7.146e-2	1.532e-1	1.580e-2	6.695e-2
-0.999	7.916e-1	2.605e-1	2.913e-1	1.226e-1	4.768e-2	2.196e-2

TABLE 4. The comparison of error terms $Qe_n(s)$ and $Me_n(s)$ for Example 3

s	$Me_n(s)$ [13]			$Qe_n(s)$(11)			
	$n=13$	$n=15$	$n=19$	$n=13$	$n=15$	$n=19$	$n=20$
0.999	6.092e-3	1.863e-2	9.825e0	3.002e-4	1.406e-5	1.385e-8	6.319e-10
0.753	6.688e-3	4.460e-2	1.083e0	9.126e-5	2.509e-7	8.004e-10	6.528e-11
0.352	6.233e-4	1.067e-2	3.697e-1	4.099e-5	3.483e-6	1.565e-9	5.362e-11
0.001	3.081e-3	4.409e-2	1.212e0	8.363e-5	3.644e-6	3.330e-9	4.196e-15
-0.001	3.111e-3	4.361e-2	1.208e0	8.363e-5	3.644e-6	3.330e-9	4.212e-15
-0.352	3.111e-3	2.288e-2	5.682e-1	4.099e-5	3.483e-6	1.565e-9	5.362e-11
-0.753	9.683e-3	5.025e-2	9.905e-1	9.126e-5	2.509e-7	8.004e-10	6.528e-11
-0.999	2.291e-3	1.083e-1	7.677e0	3.002e-4	1.406e-5	1.385e-8	6.319e-10

TABLE 5. The error comparisons between $Qe_n(s)$ and $Me_n(s)$ for lager "n"

The numerical results of Table 4 and 5 reveal that Maleknejat's proposed method is not suitable for large number of points and cannot take "n" as even number.

Furtunately error of proposed method Eq. (11) decreases very fast when number of points increses as well as can take "n" as even number.
Examples 4 (Mustafa [18]**):** For $\lambda_1 = 1$, $\lambda_2 = 1$, $a = 0$, $b = 2$ and

$$a(s) = 1, f(s) = 2\cos(s) - s\cos(2) - 2s\sin(2) + s - 1$$
$$K_1(s,t) = st, K_2(s,t) = s - t,$$

It is shown in Mustafa [18] that the exact solution of Eq. (2) is $x(s) = \cos(s)$.

For different values of "n" numerical solution are summarized in Table 6 and Table 7 for different values of "s". Let us consider Example 4 for different values of

s	$n = 5$		$n = 10$	
	$Qe_n(s)$(11)	$Mue_n(s)$ [18]	$Qe_n(s)$(11)	$Mue_n(s)$ [18]
0.0	6.117e-5	0	5.662e-11	0
0.4	9.766e-6	4.593e-6	1.379e-11	2.213e-12
0.8	4.252e-6	8.389e-6	1.037e-11	5.683e-12
1.2	3.841e-6	1.378e-5	1.059e-11	1.009e-11
1.6	7.448e-6	2.153e-5	1.470e-11	1.612e-11
2.0	3.880e-5	3.179e-5	6.304e-11	2.376e-11

TABLE 6. Comparison of error terms $Qe_n(s)$ and $Mue_n(s)$ for Example 4

"s" as shown in Mustafa [18]. From Table 6-7 we can conclude that the proposed

s	$n = 9$	
	$Qe_n(s)$(11)	$Mue_n(s)$ [18]
0.0	9.194e-10	0
0.2222	2.775e-10	5.694e-11
0.4444	2.343e-10	9.869e-11
0.6667	1.951e-10	1.478e-10
0.8889	7.917e-10	2.037e-10
1.1111	7.663e-10	2.698e-10
1.3333	1.775e-10	3.493e-10
1.5556	2.001e-10	4.458e-10
1.7778	2.224e-10	5.663e-10
2.0000	6.907e-10	7.011e-10

TABLE 7. Comparison of error terms $Qe_n(s)$ and $Mue_n(s)$ for Example 4

method is comparable with Mustafa [18] method. At certain points the proposed method yields slightly better results than the Mustafa's method and vice versa.
Examples 5 (Mustafa [18]]): Next $\lambda_1 = 1$, $\lambda_2 = 1$, $a = 0$, $b = 1$ and

$$a(s) = 1, f(s) = e^s + e^s(s-1) - se - s^2(e^s - 1) + 1$$
$$K_1(s,t) = st + s, K_2(s,t) = s^2 - t$$

The exact solution of Eq. (2) is given by $x(s) = e^s$.

For different values of "n" and "s", numerical results are summarized in Table 8 and Table 9.

s	$n = 5$	
	$Qe_n\,(s)(11)$	$Mue_n\,(s)$ [18]
0.0	2.304e-6	0
0.2	4.121e-7	1.263e-6
0.4	2.096e-7	2.555e-6
0.6	1.903e-7	3.879e-5
0.8	4.451e-7	5.506e-5
1.0	2.710e-6	7.751e-5

TABLE 8. Comparison of error terms $Qe_n\,(s)$ and $Mue_n\,(s)$ for Example 5

s	$n = 9$	
	$Qe_n\,(s)(11)$	$Mue_n\,(s)$ [18]
0.0	9.194e-10	0
0.1111	2.775e-10	9.133e-13
0.2222	2.343e-10	1.842e-12
0.3333	1.951e-10	2.753e-12
0.4444	7.917e-10	3.678e-12
0.5556	7.663e-10	4.638e-12
0.6667	1.775e-10	5.685e-12
0.7778	2.001e-10	6.871e-12
0.8889	2.224e-10	8.292e-12
1.0	6.907e-10	1.005e-11

TABLE 9. Comparison of error terms $Qe_n\,(s)$ and $Mue_n\,(s)$ for Example 5

As we can see in Table 8 and Table 9, the proposed method is comparable with Mustafa's method. The proposed method shown better result than Mustafa's method for "$n = 5$" while Mustafa's method got better for "$n = 9$".

6. **Conclusion.** In this work, we have used Gauss-Legendre QF and reduction technique to solve Eq. (2) on the interval $[a,\ b]$. Efficient method is presented to solve the linear IEs of the third kind. Moreover, we have compared our results with Chakrabarti [6], Maleknejad [13] and Mustafa [18], for the same examples with the same number of points. All the Tables show that proposed method is comparable with other methods and in all cases the error of suggested method decreases when the number of points increase. All numerical calculations are made by Maple 17.

Acknowledgements. This work was supported by Universiti Sains Islam Malaysia (USIM) under Research Grand PPP/GP/FST/30/14915, and University Putra Malaysia under Fundamental Research Grant Scheme (FRGS). Project code is 01-12-10-989FR.

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Received March 2016; 1^{st} revision May 2016; final revision February 2017.

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